Outline

- Introduction
- Korteweg-de Vries equations
- Zakharov-Kuznetsov in 2D
- Zakharov-Kuznetsov in 3D
- Outlook

Korteweg-de Vries equation

- Boussinesq 1877, Korteweg-de Vries 1895

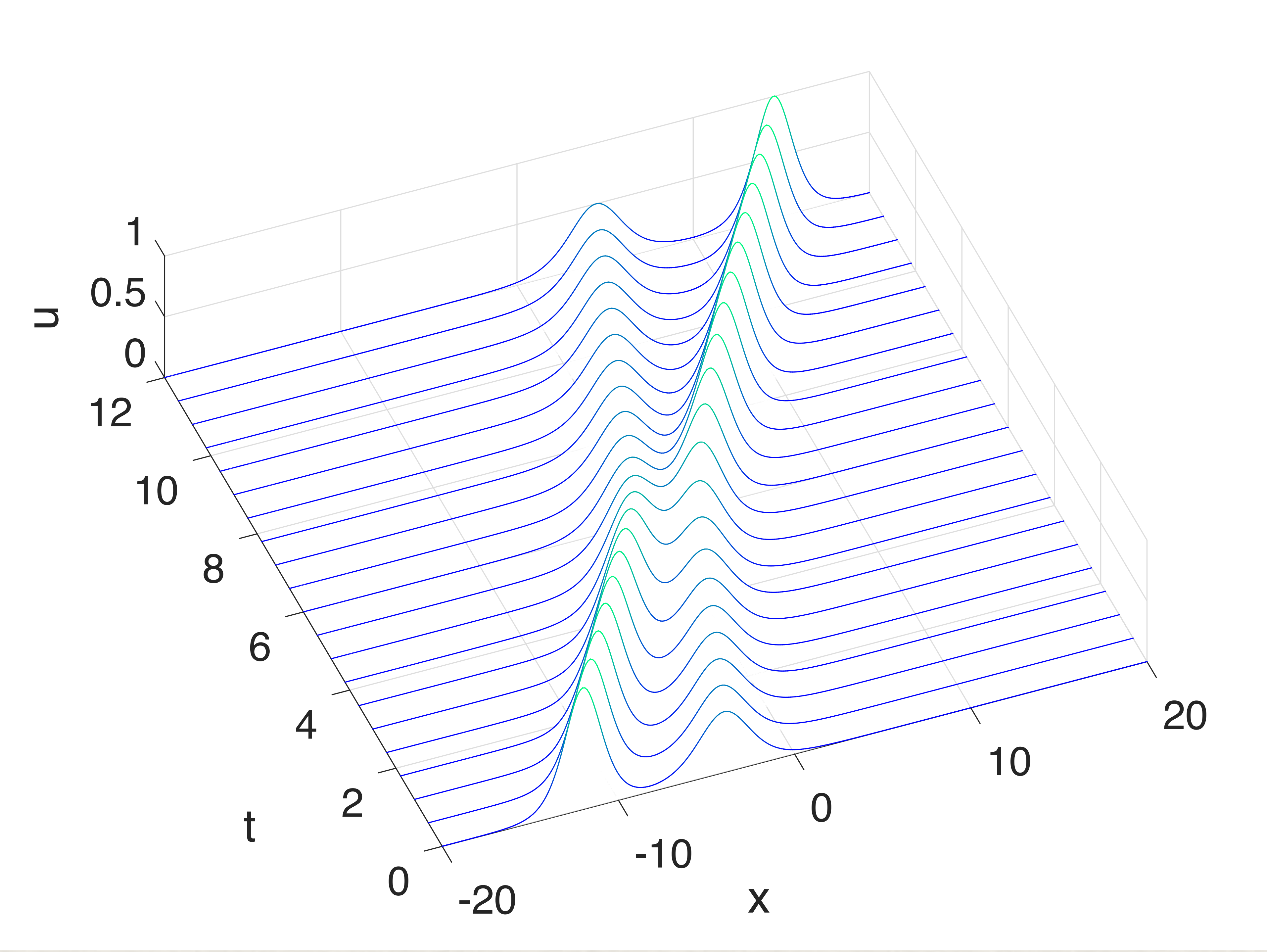
$$u_t + 6uu_x + u_{xxx} = 0$$

- 1-soliton solution (solitary travelling wave):

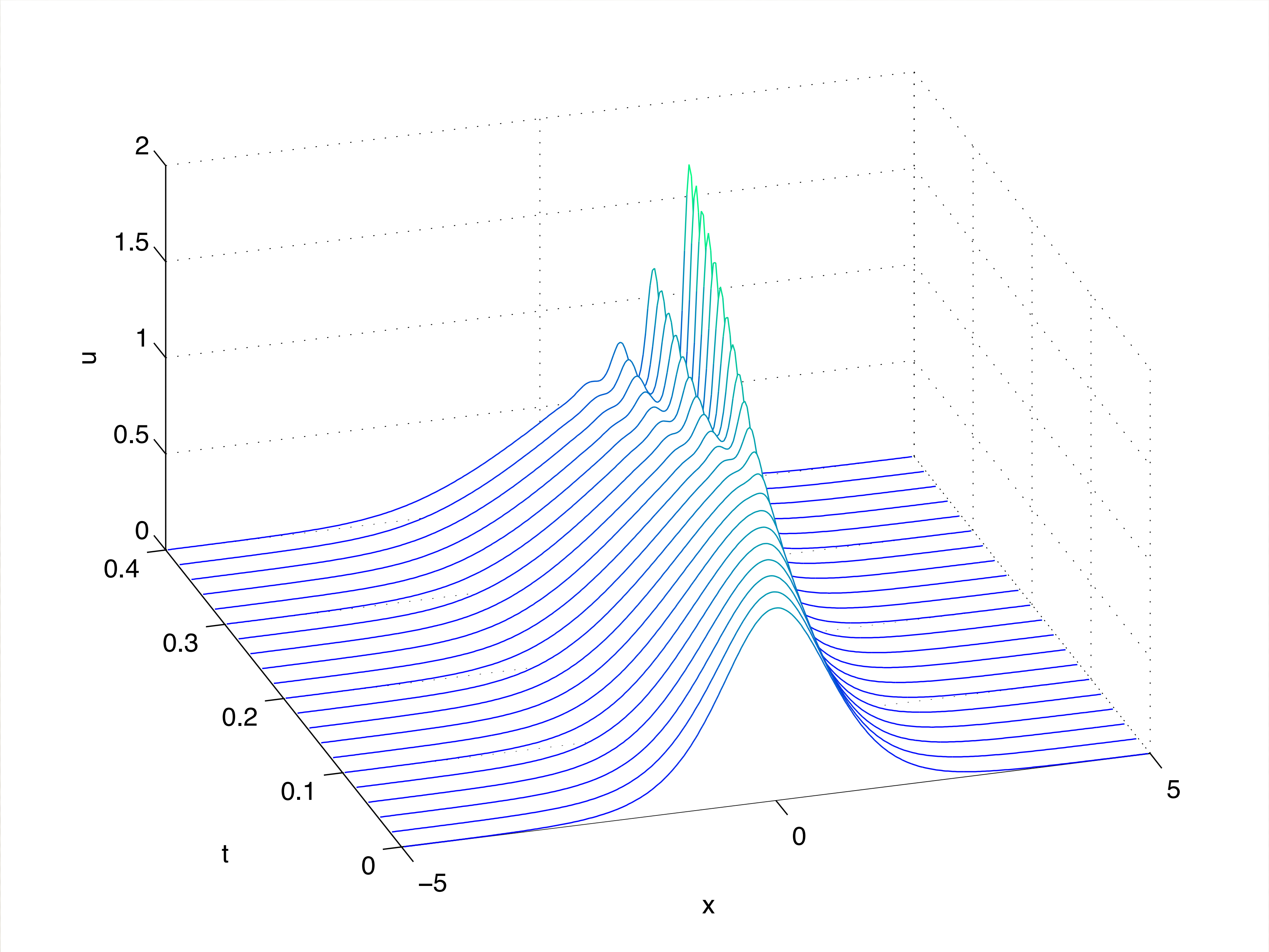
$$u(x,t) = \frac{c}{2}\text{sech}^2(\sqrt{c}(x-ct)), \quad c = const$$

- 1967: Gardener, Greene, Kruskal and Miura: KdV equation is completely integrable

2-soliton solution



Soliton resolution



$$u_t + 6uu_x + \epsilon^2 u_{xxx} = 0, \quad u(x,0) = \operatorname{sech}^2 x$$
$$(x \mapsto \epsilon x, \quad t \mapsto \epsilon t)$$

Blow-up

- generalized KdV equation

$$u_t + u^p u_x + \epsilon^2 u_{xxx} = 0, \quad p \in \mathbb{N}$$

- $$u_t + \epsilon^2 u_{xxx} = 0$$

  (linear) and

  $$u_t + u^p u_x = 0$$

  (shocks) do not have blow-up of the L_∞ norm of u

- for $p < 4$: global existence in time,
  for $p = 4$: finite time blow-up (Martel, Merle, Raphaël: rescaled soliton),
  for $p > 4$: finite time blow-up, no theory yet.

Theorem: Martel, Merle, Raphaël (2013)
Let $\mathcal{T}_{\alpha^*}$ be the set given by

$$\mathcal{T}_{\alpha^*} = \left\{ u \in H^1 \text{ with } \inf_{\lambda_0>0, x_0\in\mathbb{R}} \left|\left| u - \frac{1}{\lambda_0} Q\left(\frac{\cdot - x_0}{\lambda_0}\right)\right|\right|_2 < \alpha^* \right\}$$

and let $\mathcal{A}$ be the set of initial data $u(x,0) = u_0$ given by

$$\mathcal{A} = \left\{ u_0 = Q + \epsilon_0 \text{ with } ||\epsilon_0||_{H^1} < \alpha_0 \ll 1 \text{ and } \int_{x>1} x^{10}\epsilon_0^2 dx < \infty \right\}, \qquad (1)$$

where $0 < \alpha_0 \ll \alpha^* \ll 1$ are universal constants, and let $u_0 \in \mathcal{A}$. If $E[u_0] < 0$, then $u(x,t)$ blows up at the finite time t^* and $u(t) \in \mathcal{T}_{\alpha^*}$ for $t < t^*$. Then there exist a constant (with respect to t) $l_0(u_0) > 0$ and functions $L(t)$ and $x_m(t)$ such that for $t \to t^*$

$$u(x,t) - \frac{1}{\sqrt{L(t)}} Q\left(\frac{x - x_m(t)}{L(t)}\right) \to \tilde{u} \in L_2, \qquad (2)$$

and

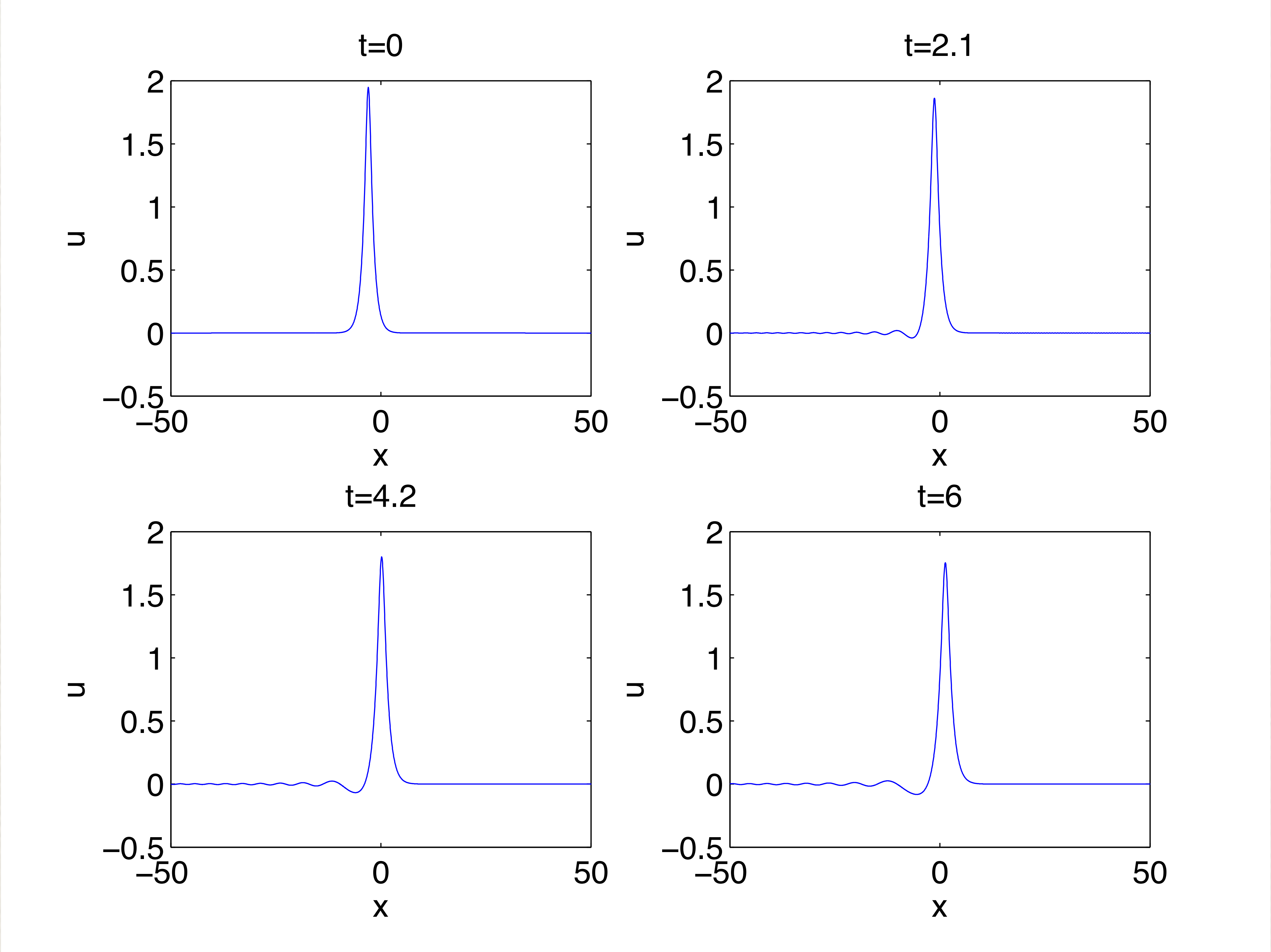
$$||u_x||_2 \sim \frac{||Q'||_2}{l_0(t^* - t)} \qquad (3)$$

with Q from (**??**), where

$$L(t) \sim l_0(t^* - t), \quad x_m(t) \sim \frac{1}{l_0^2(t^* - t)}. \qquad (4)$$

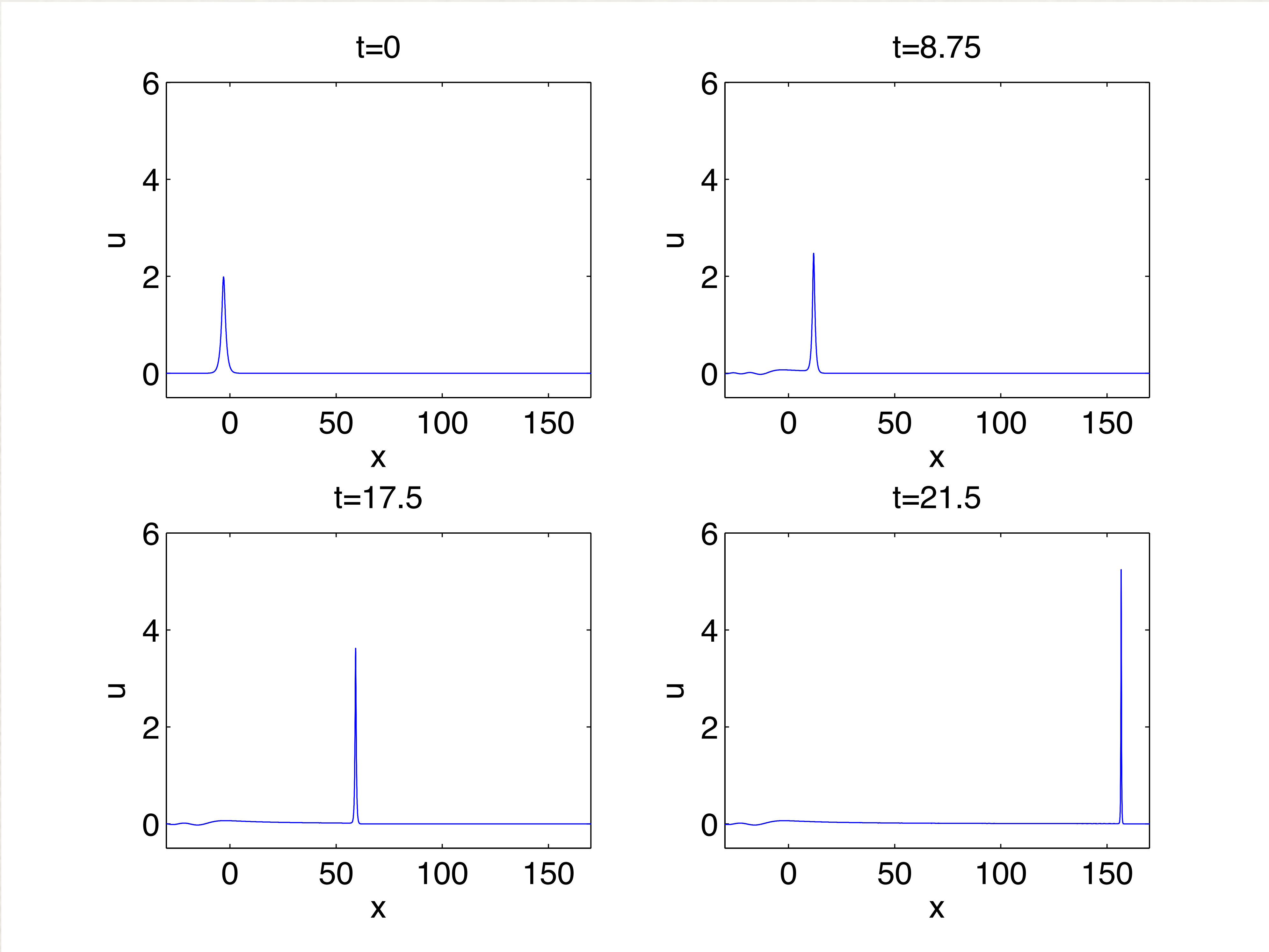
Perturbed gKdV soliton, n=4

$u_0 = 0.99u_{sol}$



Perturbed solitary wave, n=4

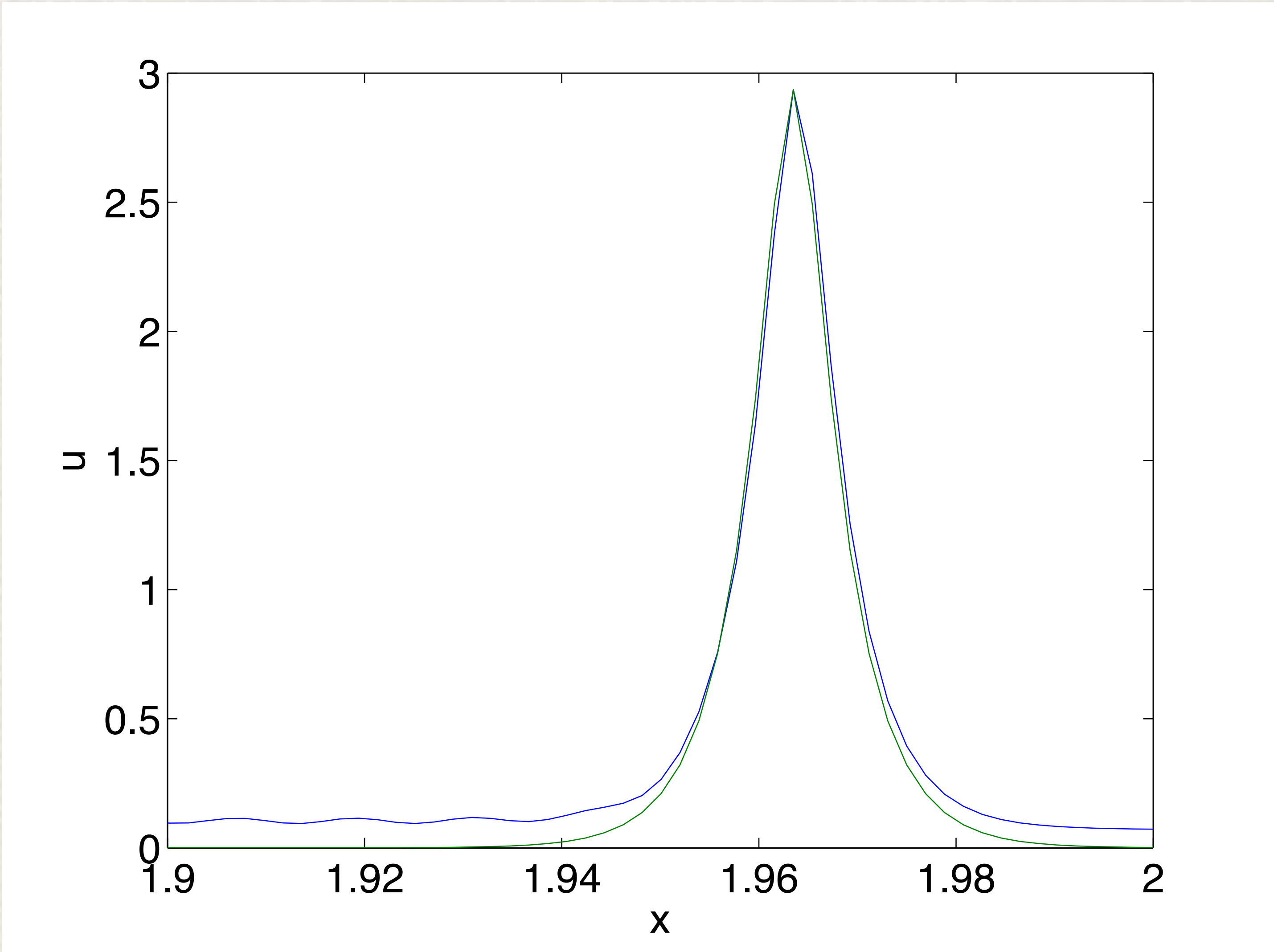
$$u_0 = 1.01u_{sol}$$



Fitting to rescaled soliton

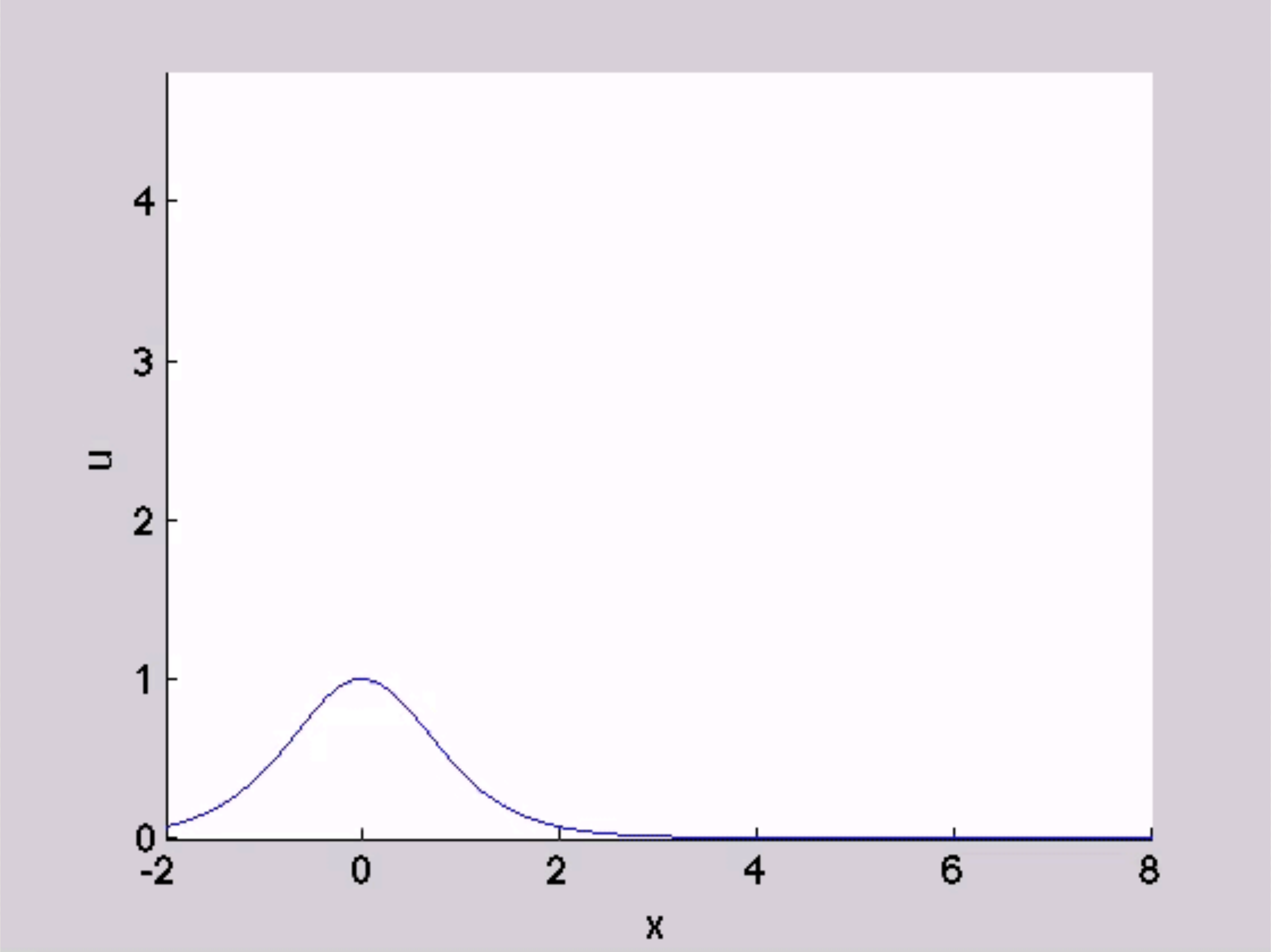
- Martel, Merle, Raphaël 2012: selfsimilar blow-up, blow-up profile dynamically rescaled soliton

C. Klein and R. Peter, *Numerical study of blow-up in solutions to generalized Korteweg-de Vries equations*, Physica D 304-305 (2015), 52-78



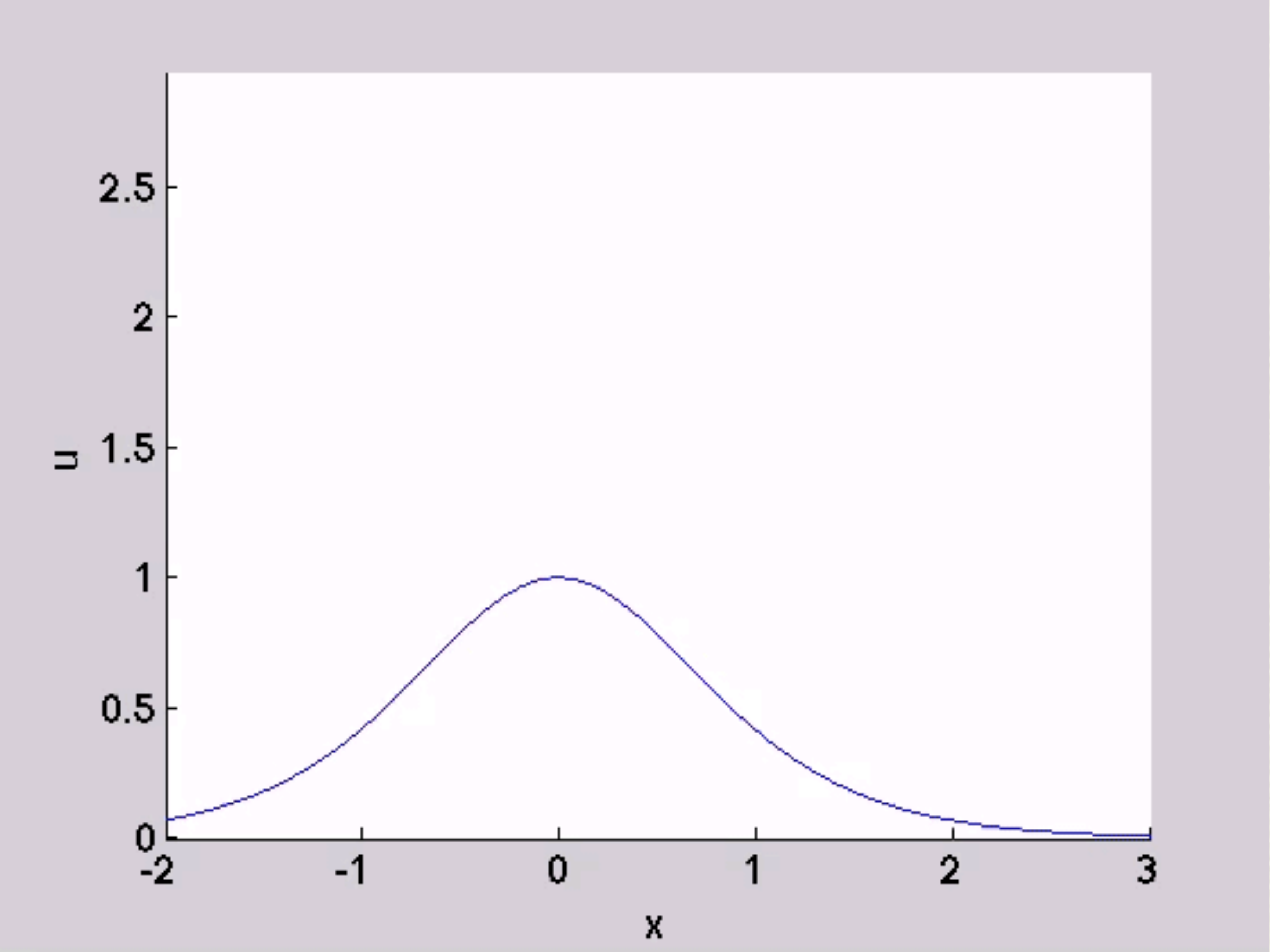
gKdV, small dispersion

$$u_0 = \mathrm{sech}^2 x, \quad \epsilon = 0.1 \qquad n = 4$$



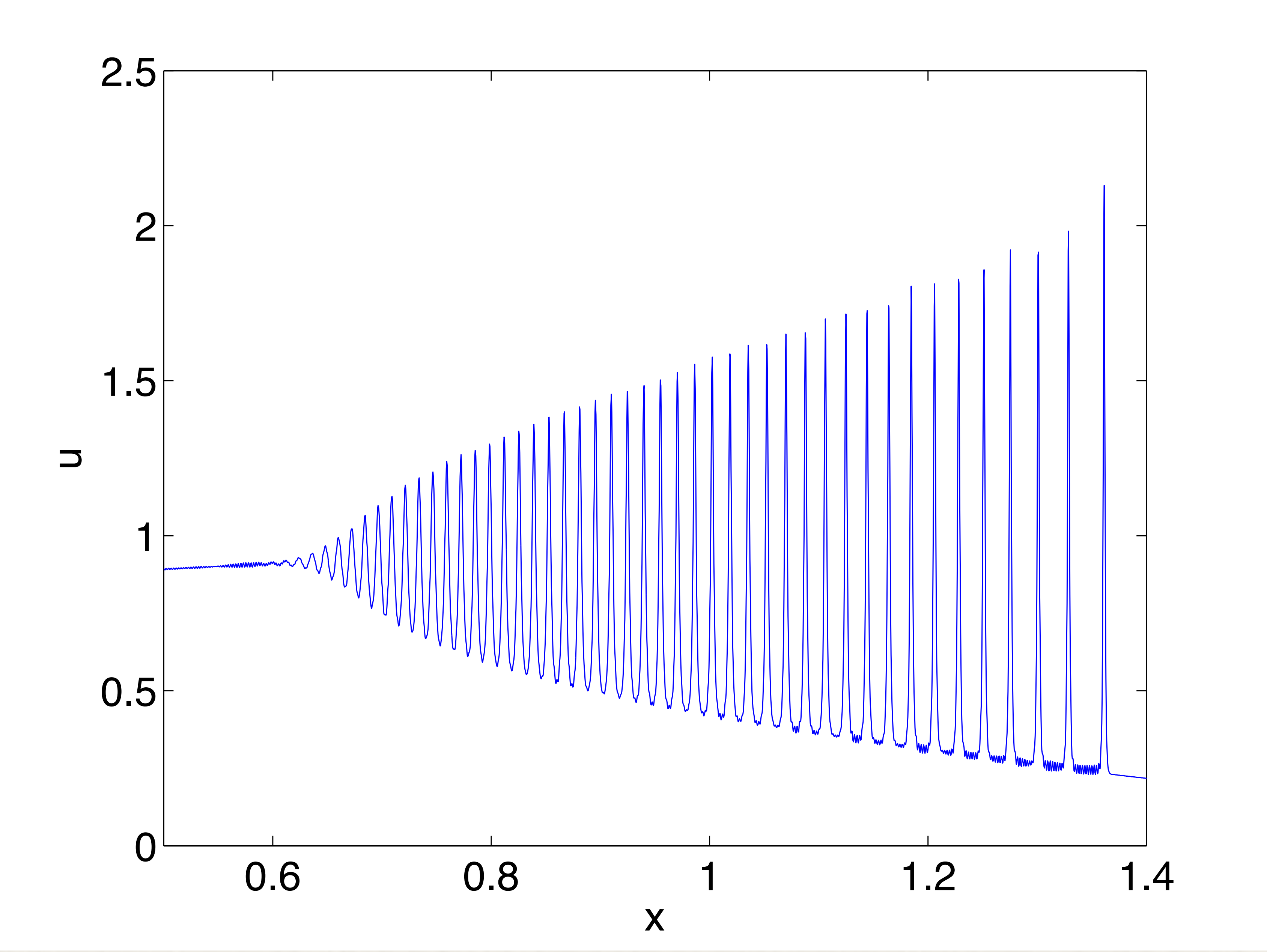
gKdV, small dispersion

$$u_0 = \text{sech}^2 x, \quad \epsilon = 0.01 \qquad n = 4$$



gKdV, small dispersion

$$u_0 = \operatorname{sech}^2 x, \quad \epsilon = 0.001 \qquad n = 4$$



2D Zakharov-Kuznetsov

K, Roudenko, Stoilov 2019

$$u_t + (\Delta u + u^p)_x = 0, \quad p = 2, 3, 4.$$

- Introduced by Zakharov and Kuznetsov in 1972 for 3D with $p = 2$:
- Describes propagation of ionic-acoustic waves in uniformly magnetized plasma
- Rigorous derivation of ZK from Euler-Poisson system with magnetic field in the long wave limit (Lannes, Linares, Saut 13)
- The amplitude equation for 2d long waves on the free surface of a thin film flowing down the vertical plane with moderate surface tension and large viscosity (Melkonian and Maslowe 89)
- Derivation of ZK from Vlasov-Poisson system in a combined cold ions and long wave limit (Han-Kwan 13)

K, Saut, Stoilov 2023

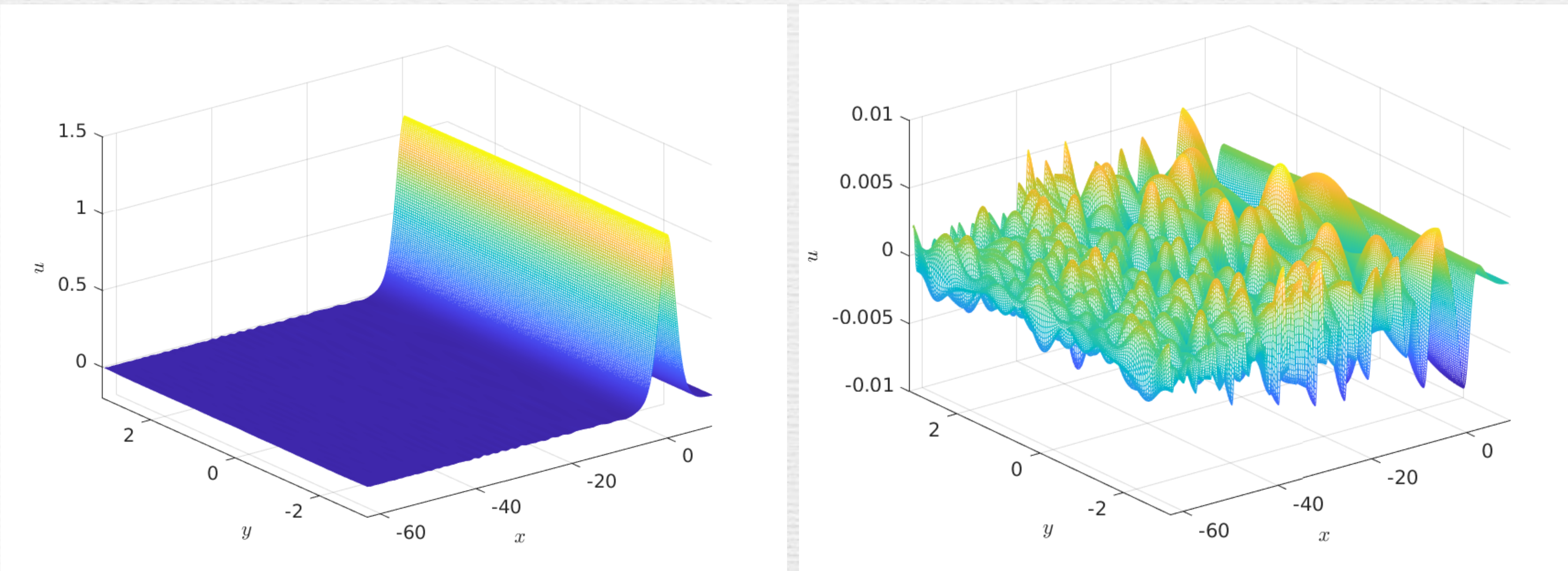
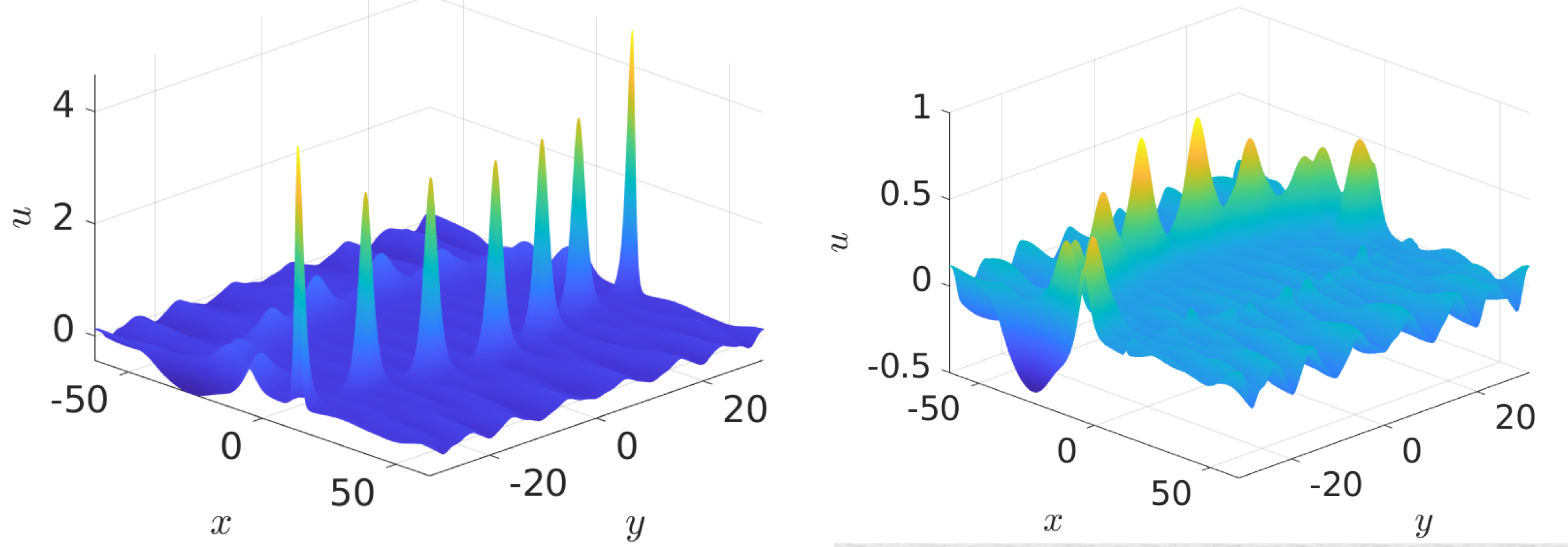


FIGURE 2. Solution to the subcritical ($p = 2$) ZK for locally perturbed soliton initial data (17) and sub-critical speed: on the left the solution for $t = 1$ and on the right the difference between the final state and a fitted line soliton (2).



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Soliton stability

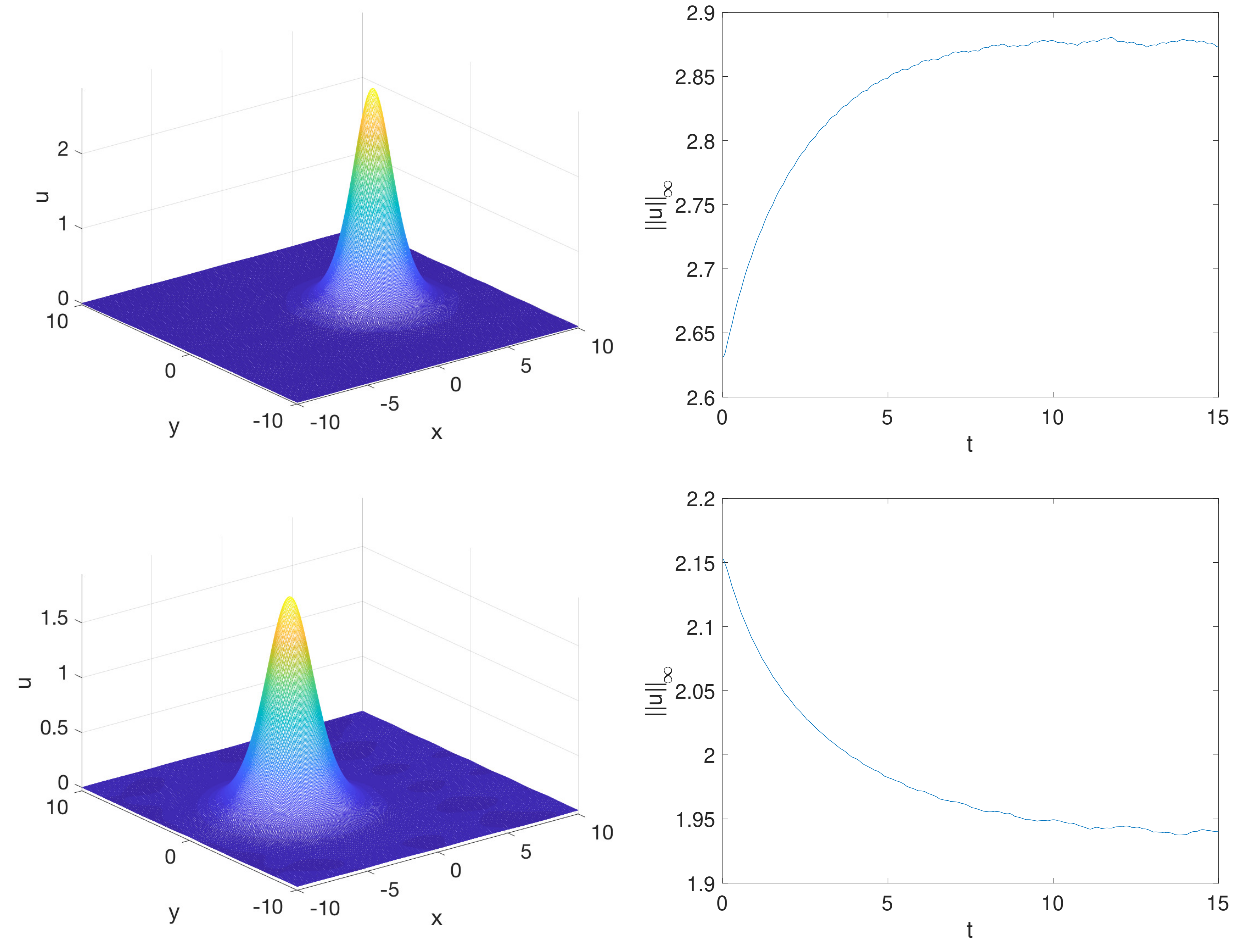


Figure: top: Solution to the ZK equation $u(x, y, 0) = 1.1Q(x, y)$: on the left the solution for $t = 15$, and on the right the L^∞ norm. Bottom: Solution for $u(x, y, 0) = 0.9Q(x, y)$: on the left the solution for $t = 10$, and on the right the L^∞ norm.

Asymptotic profile

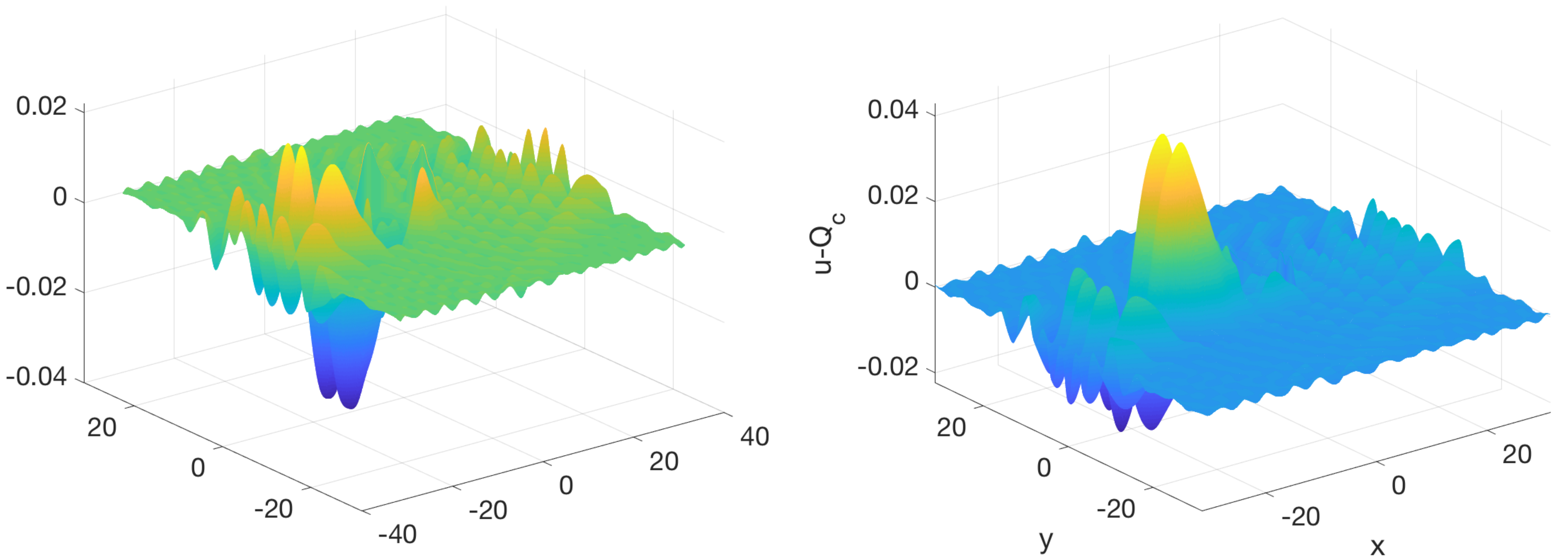
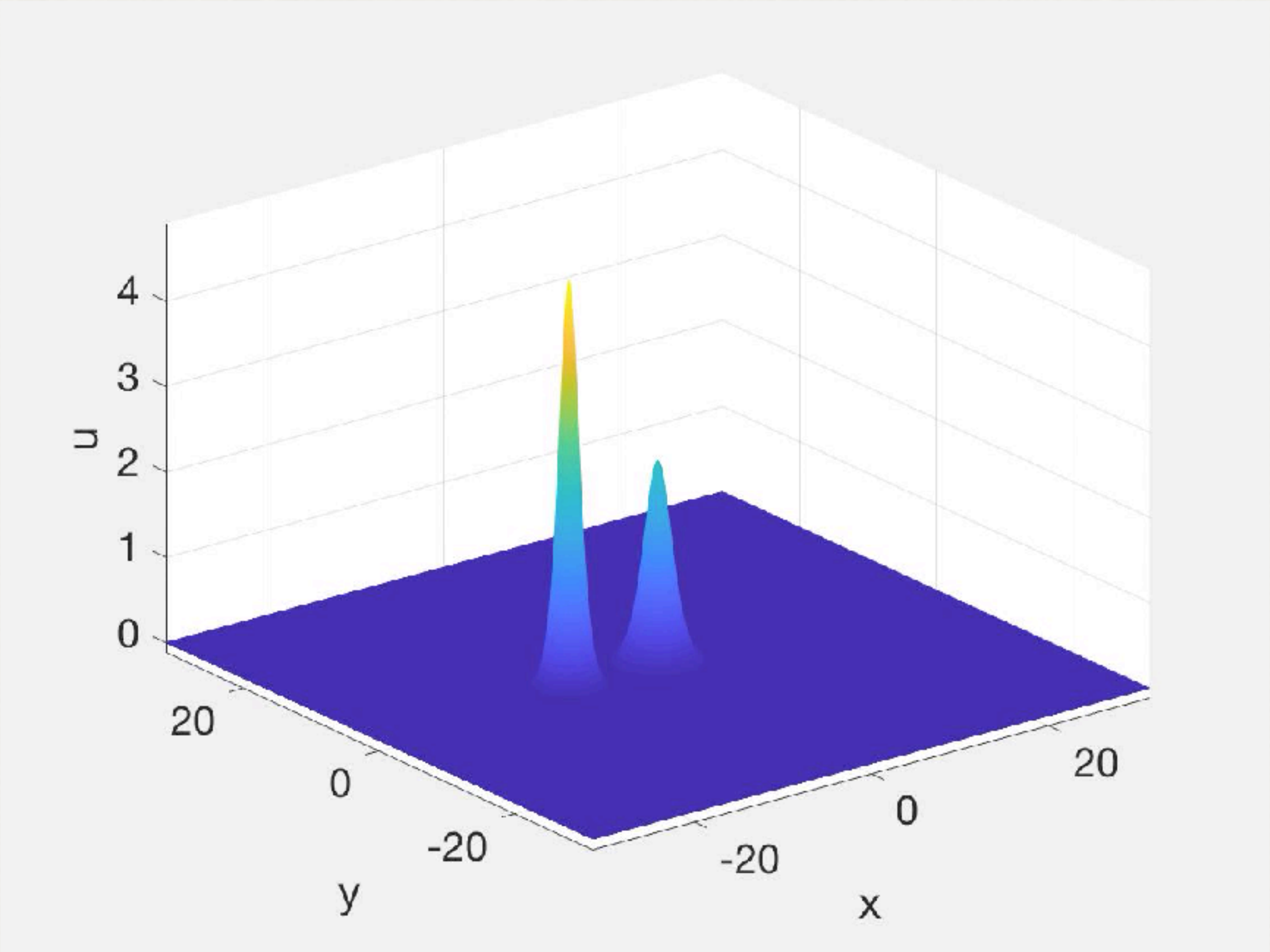


Figure: Difference of the solution to the ZK equation for the initial data $u(x, y, 0) = \lambda Q(x, y)$ and a fitted rescaled soliton : on the left $\lambda = 0.9$, on the right for $\lambda = 1.1$.

Soliton interaction (on the x-axis)



Soliton Interactions

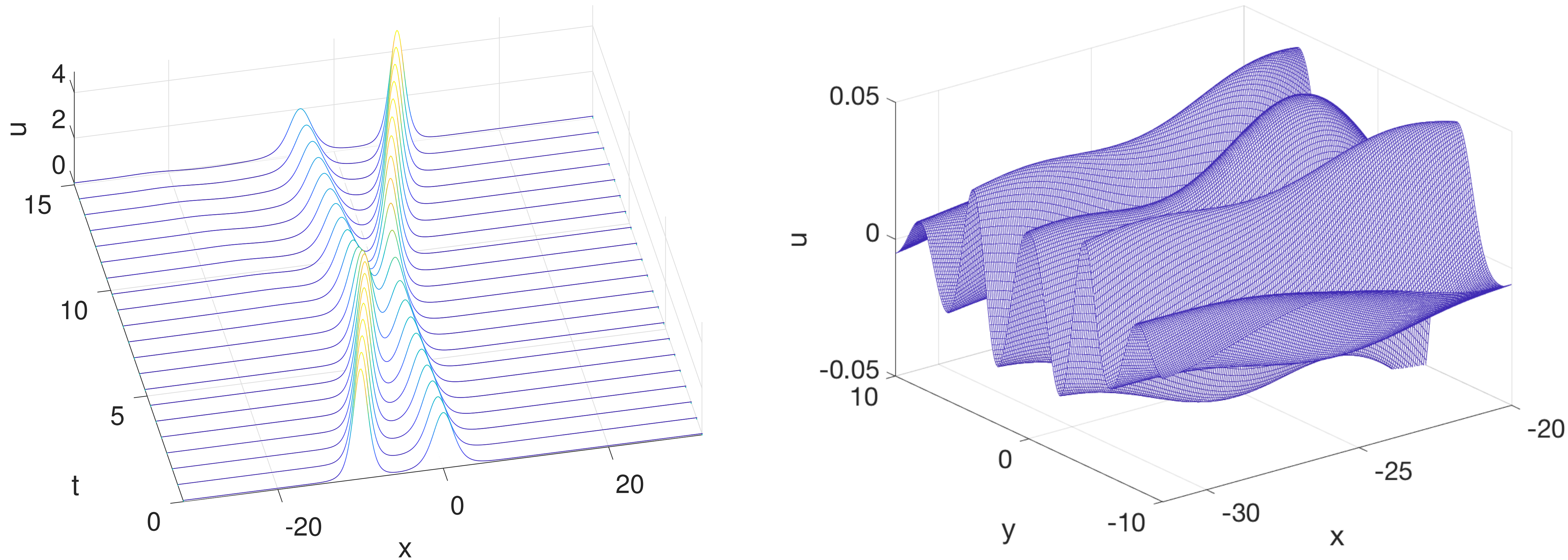
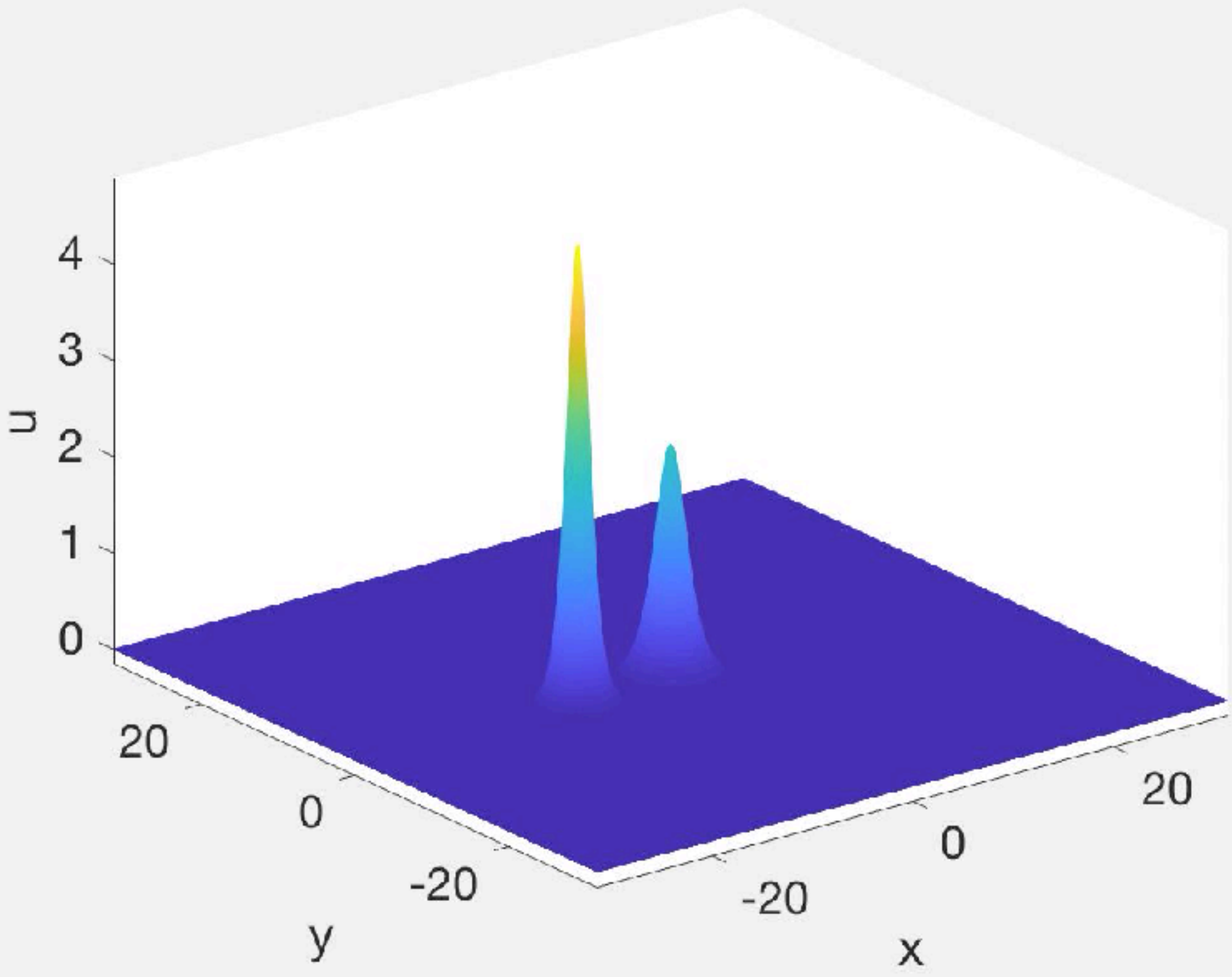
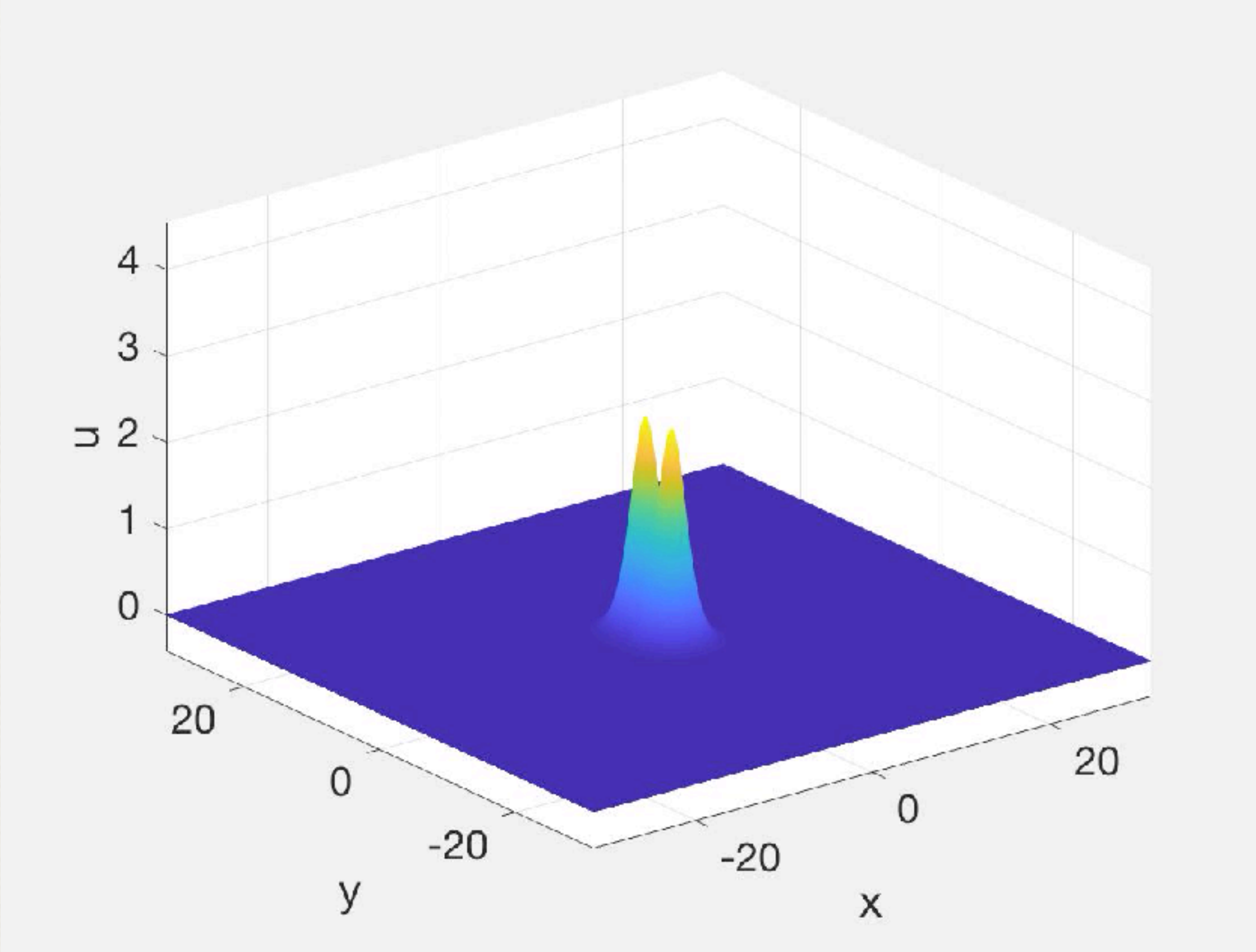


Figure: Solution to the ZK equation for initial data being the superposition of a soliton with $c = 2$ centered at $x = -10$ and a soliton with $c = 1$ centered at the origin on the x axis for various times on the left, and a close-up of the bottom right figure of the previous figure on the right.

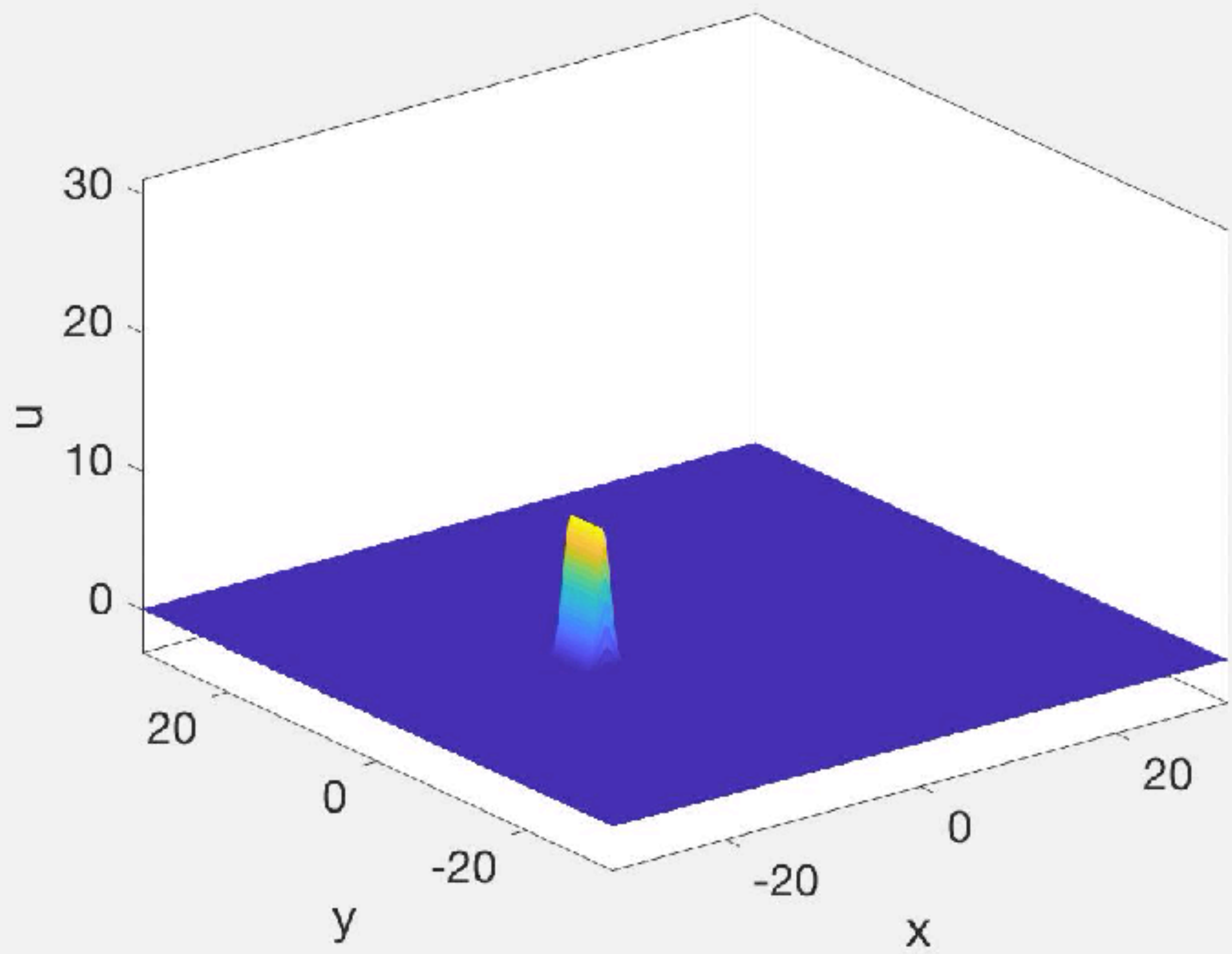
Soliton interaction (displaced)



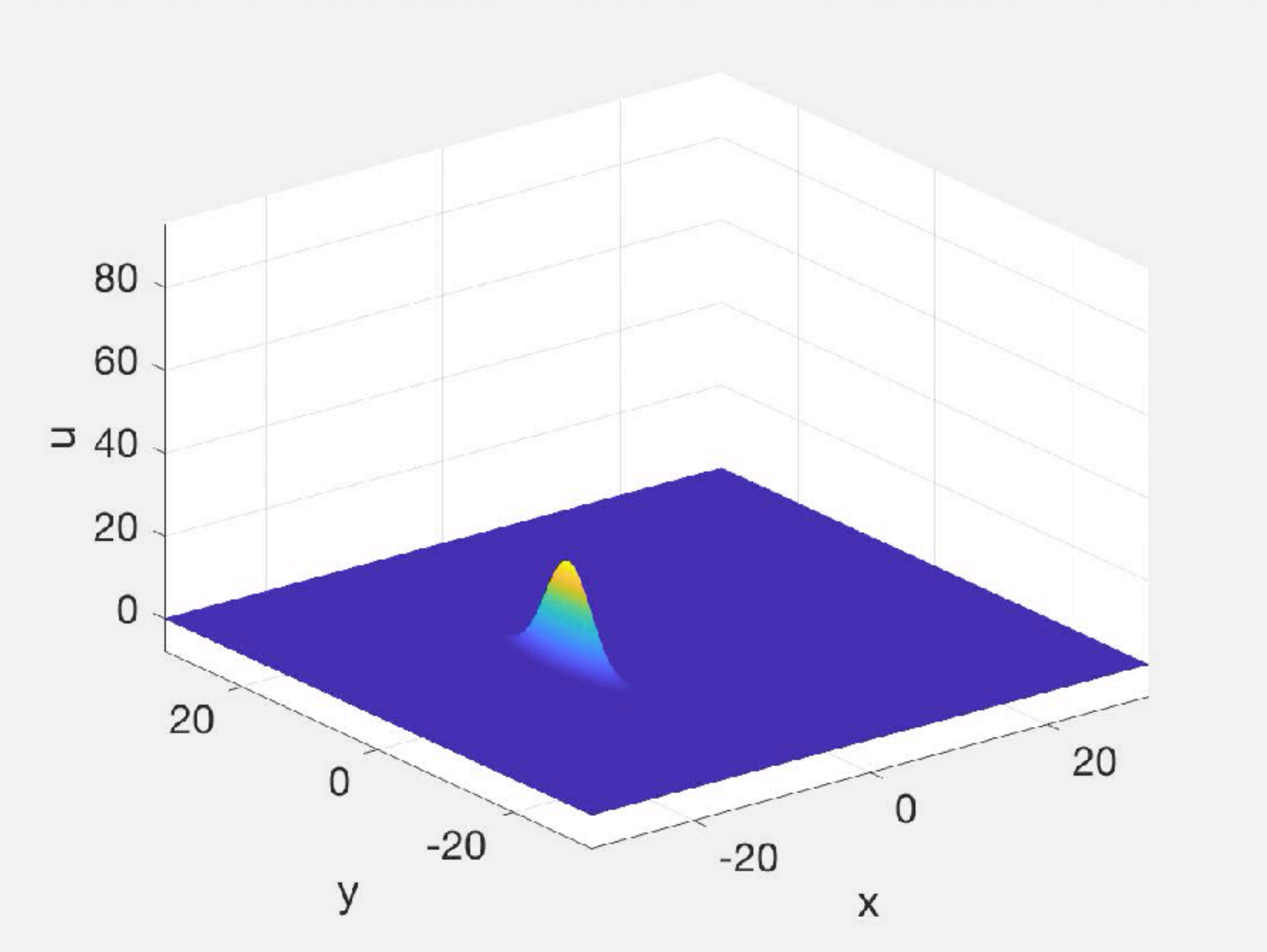
Soliton interaction (displaced)



$$u(x,y,0) = \begin{cases} 10\exp(-x^2) & |x| \le 1.5 \\ 10\exp(-x^2 - (y-1.5)^8) & x > 1.5 \\ 10\exp(-x^2 - (y+1.5)^8) & x < -1.5 \end{cases}$$



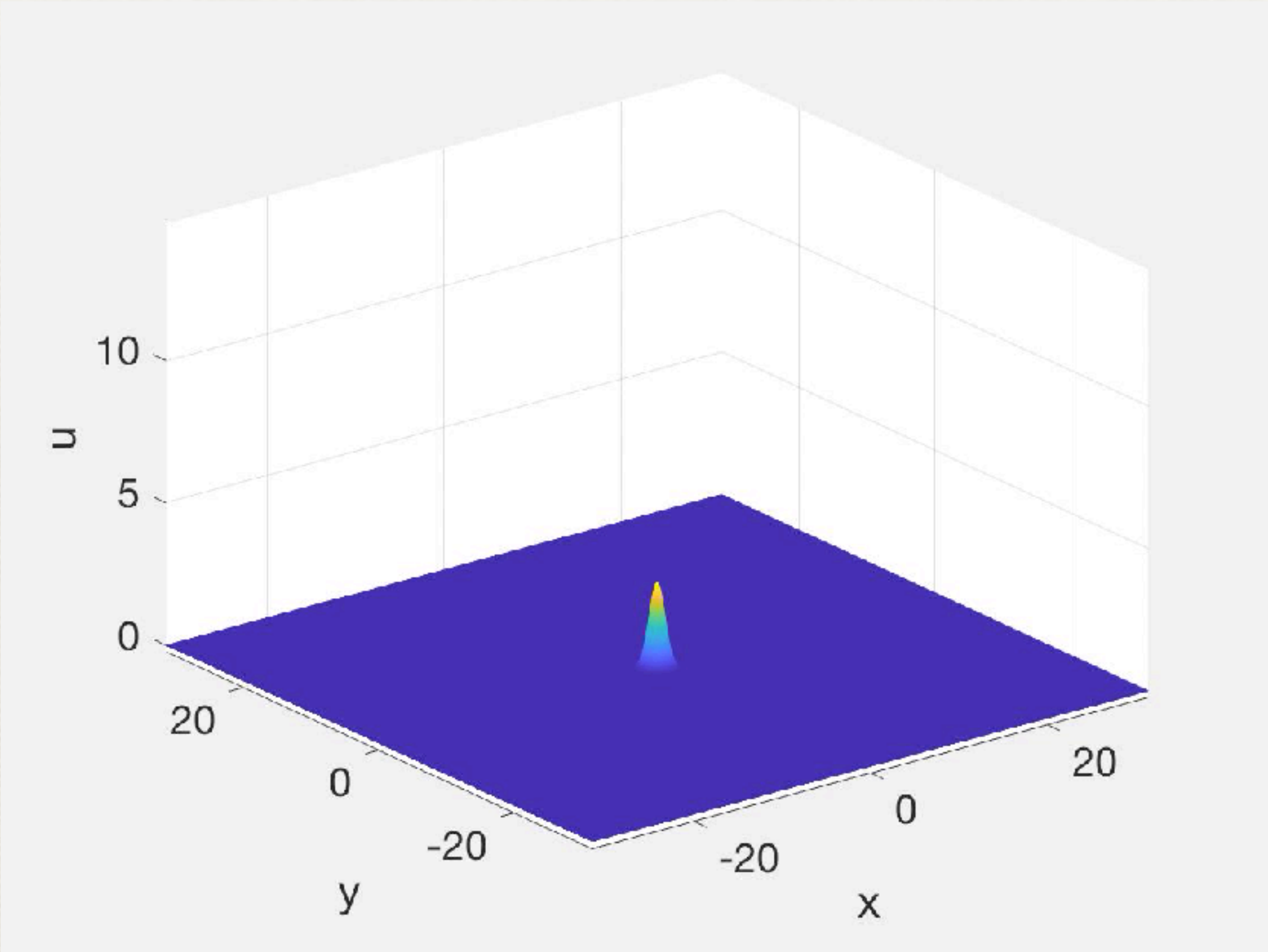
$$u(x, y, 0) = 25\exp(-x^2 - 0.05y^2)$$



K, Roudenko, Stoilov, 2019

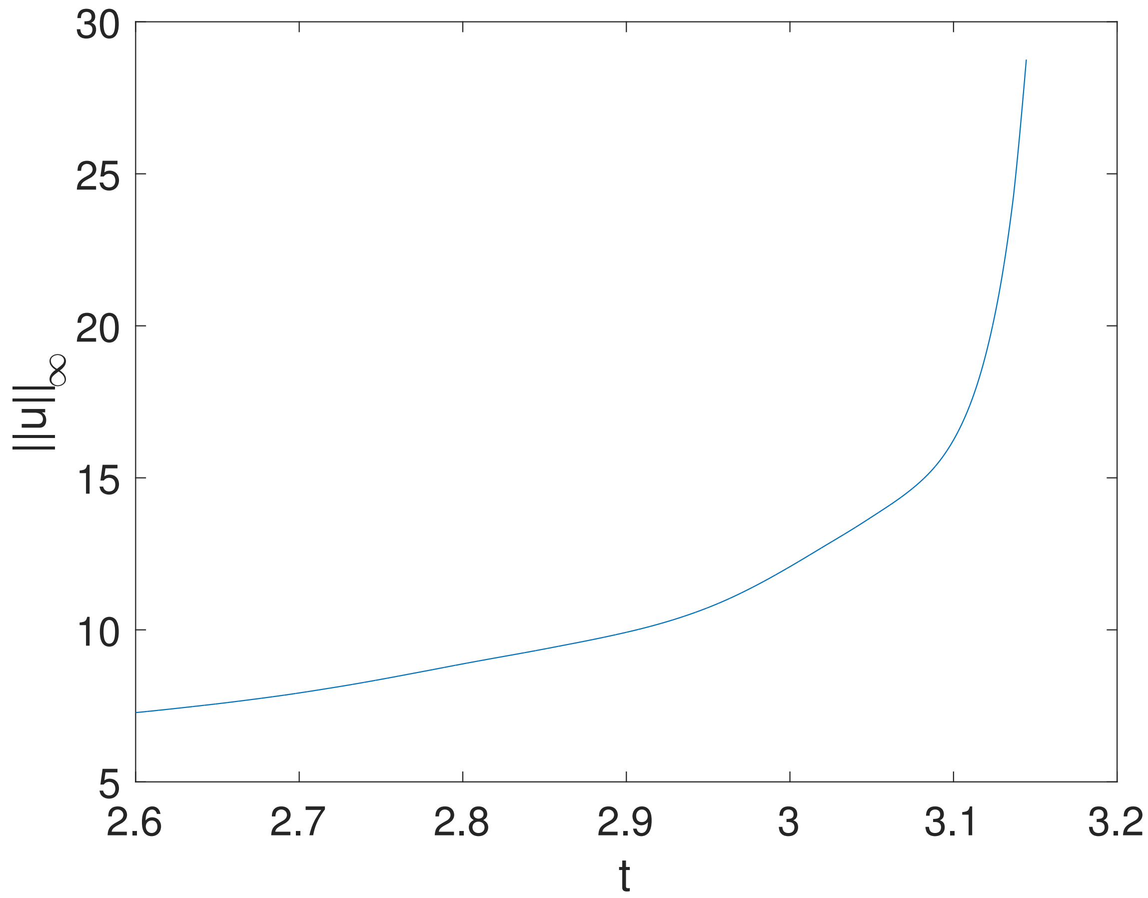
ZK with p=3

$u(x, y, 0) = 1.1Q(x, y)$



L_2 critical case.

The L^∞ norm of the solution as well as the L^2 norm of u_x indicate a blow-up.



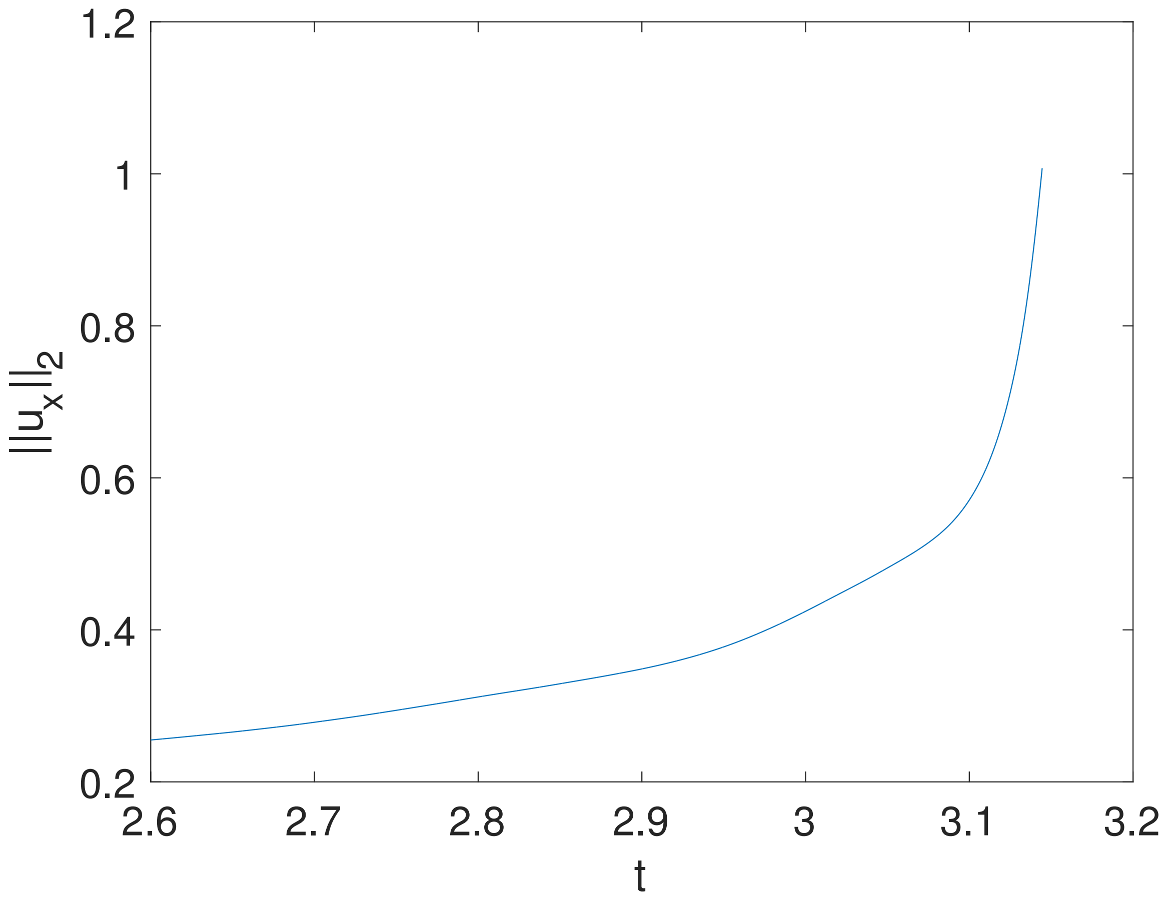


Figure: Solution to the ZK equation for the initial data $u(x, y, 0) = 1.1Q(x, y)$: on the left the L^∞ norm of the solution, on the right the L^2 norm of u_x in dependence of time.

Blow-up behaviour

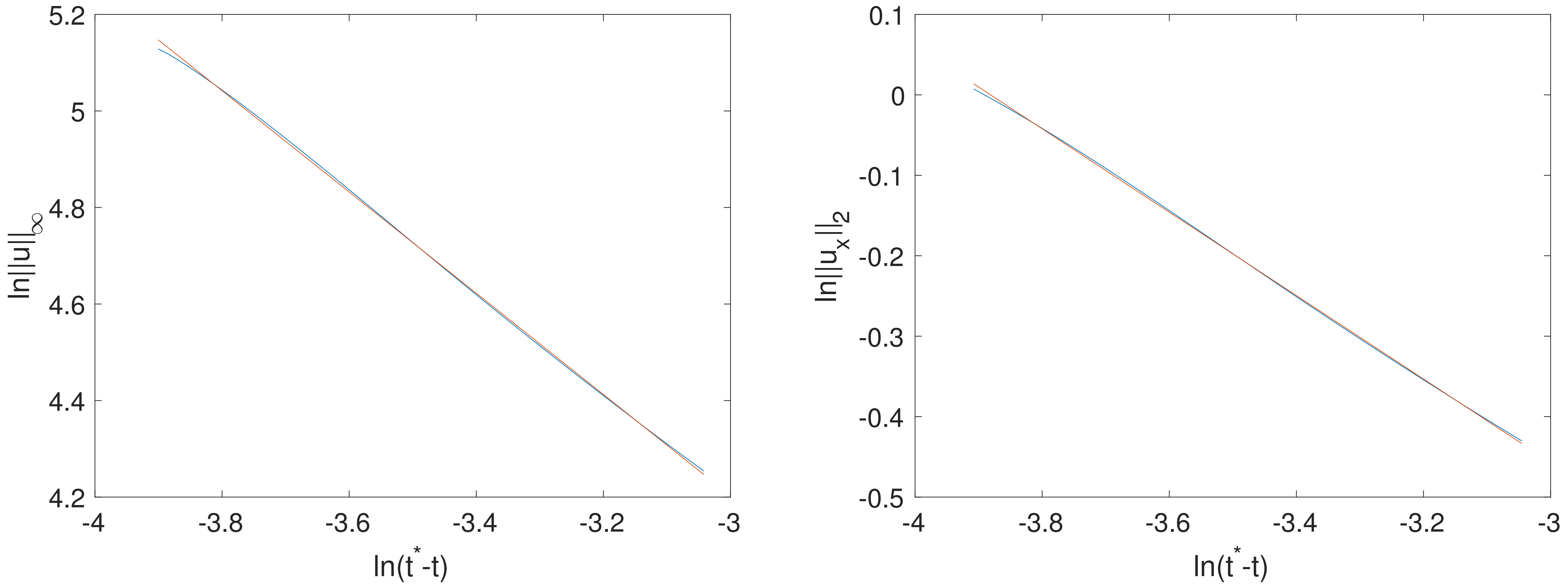


Figure: Fitting of various norms of the solution to the ZK equation for the initial data $u(x, y, 0) = 1.1Q(x, y)$ to $\ln g(t) \sim a \ln(t^* - t) + b$: on the left the L^∞ norm of the solution, on the right the L^2 norm of u_x; in red the fitted line. We get $a = -0.5185$, $b = -2.0124$ and $t^* = 0.5646$,

L_2 critical case: Conjecture

Conjecture

If $u(x, y, 0) \in \mathbb{S}(\mathbb{R}^2)$ is such that $||u(x, y, 0)||_2 < ||Q||_2$, then the solution to the ZK equation is dispersed.
If $u(x, y, 0) \in \mathbb{S}(\mathbb{R}^2)$ is such that $||u(x, y, 0)||_2 > ||Q||_2$, then the solution has a blow-up in finite time $t = t^*$ such that for $t \to t^*$

$$u(x, y, t) - \frac{1}{L(t)} Q\left(\frac{x - x_m(t)}{L(t)}\right) \to \tilde{u} \in L^2,$$

and

$$||u_x||_2 \sim \frac{1}{L}$$

where

$$L(t) \sim \sqrt{t^* - t}, \quad x_m(t) \sim \frac{1}{t^* - t}.$$

- 3D equation

$$u_t + (u_{xx} + u_{yy} + u_{zz} + u^p)_x = 0,$$

- $p = 2$: subcritical
- $p = 7/3$: L^2 critical
- $p > 7/3$: supercritical
- solitary wave: $u(x, y, z, t) = Q(x - ct, y, z)$

$$-\Delta_{\mathbb{R}^3} Q + Q - Q^2 = 0,$$

NLS solitary wave in 3D

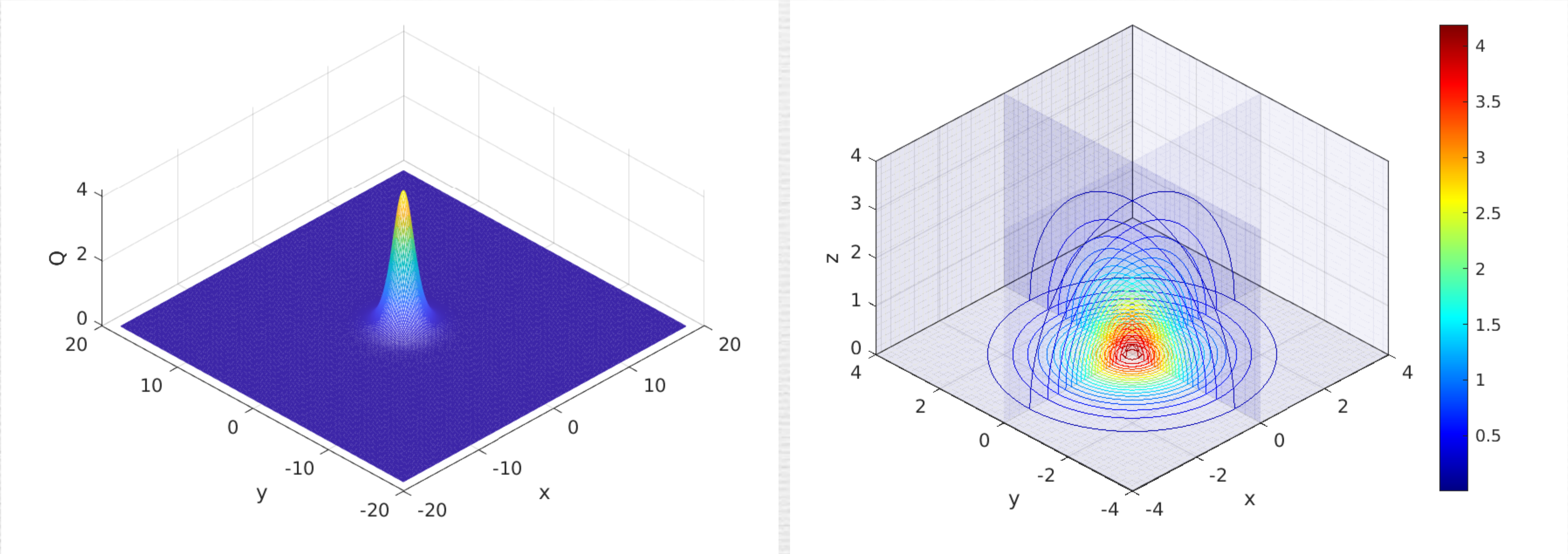
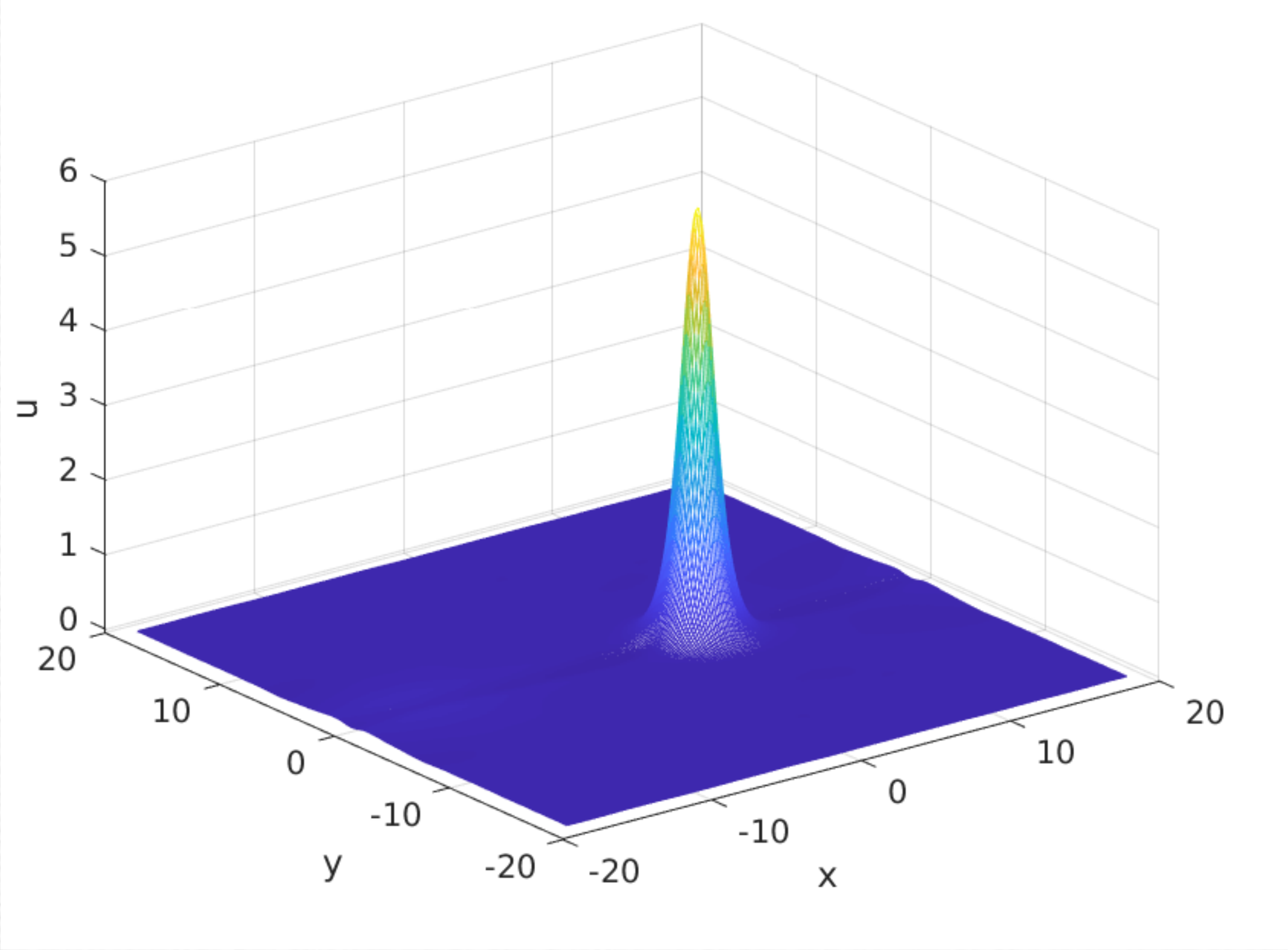
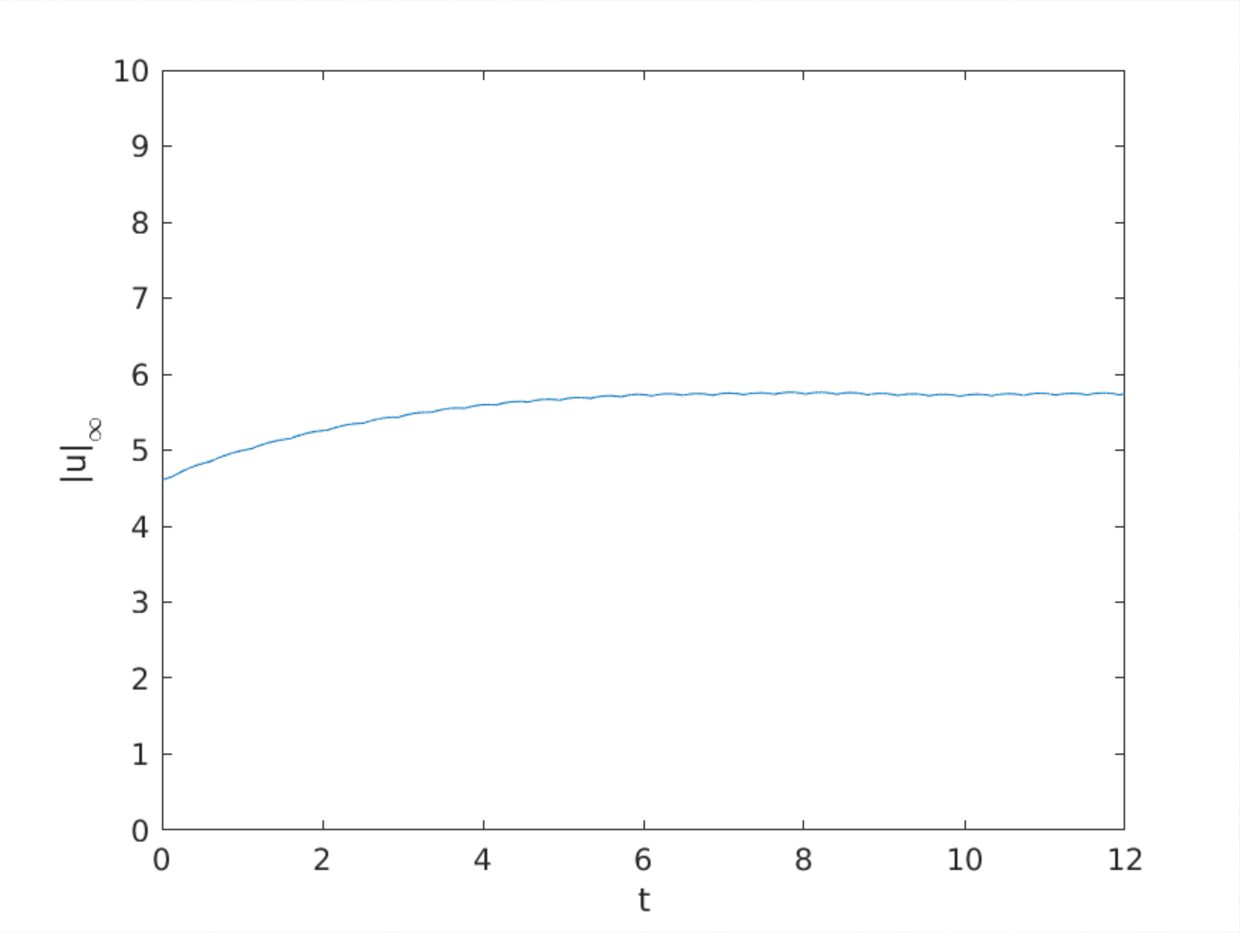
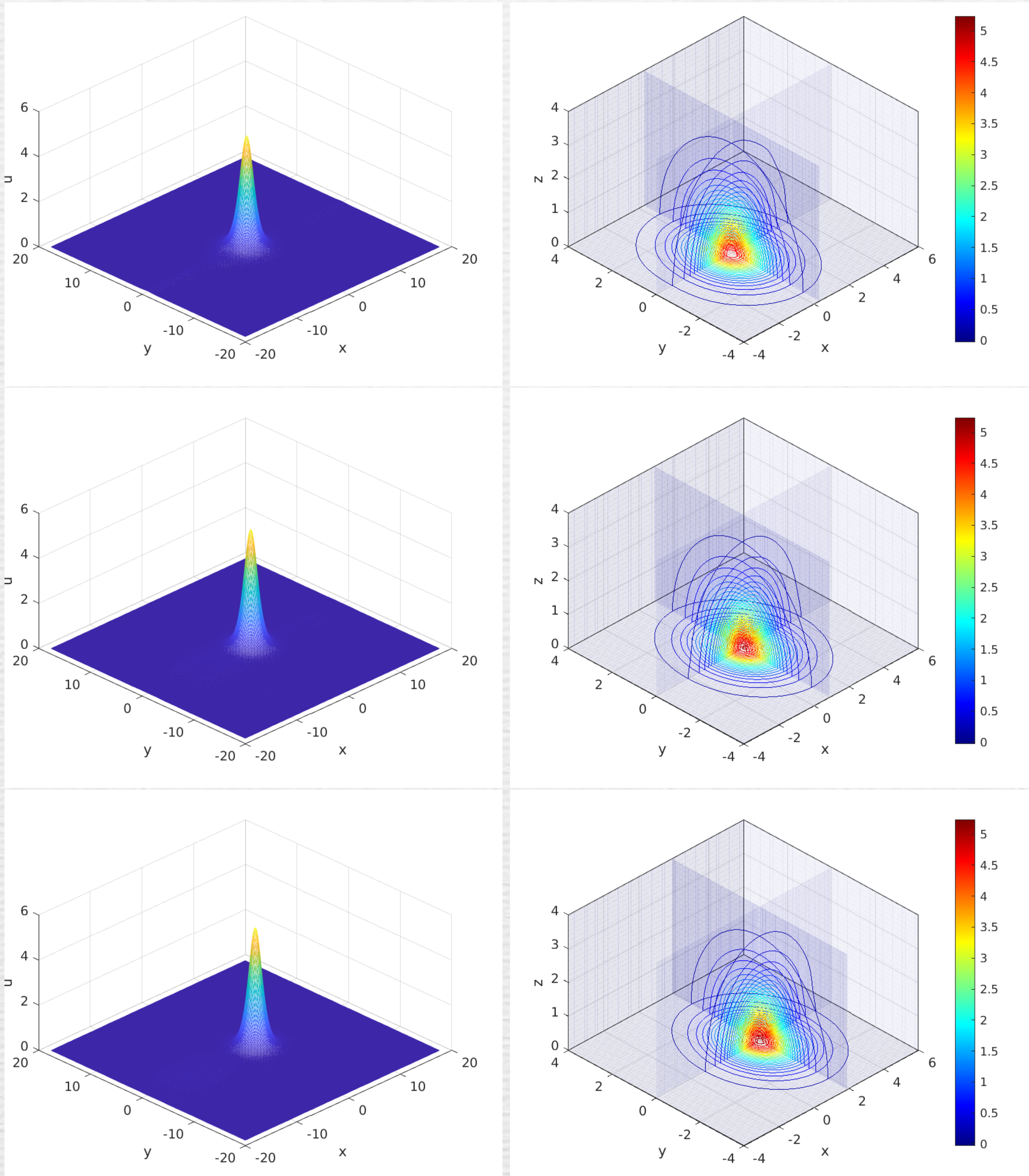


FIGURE 1. The ground state solution to (8). Left: plot of Q with $z = 0$. Right: 3D contour plots of Q on the slices of the coordinate planes. The color bar indicates the magnitude of the solution.

$$u(x, y, z, 0) = 1.1 * Q$$



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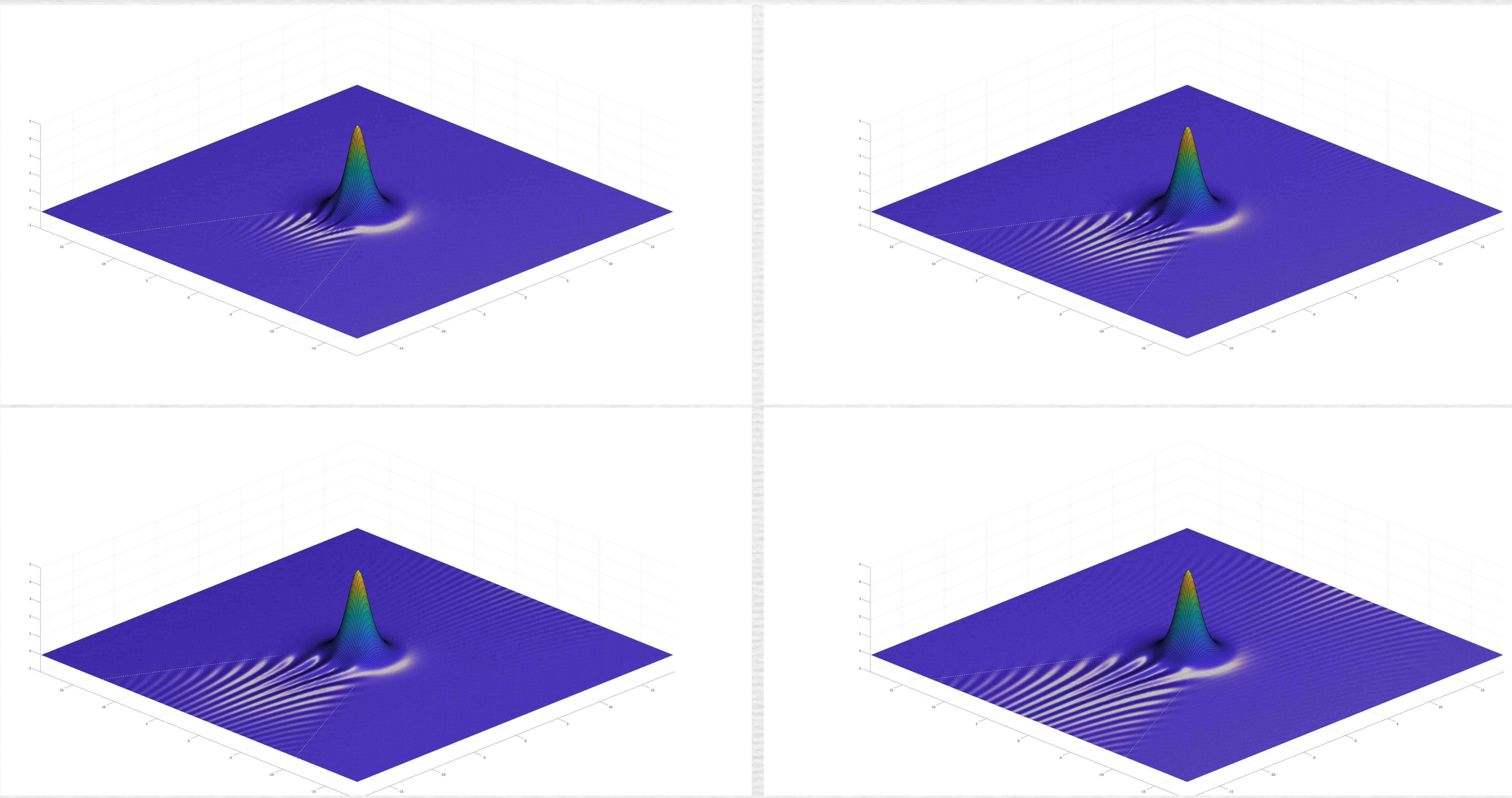
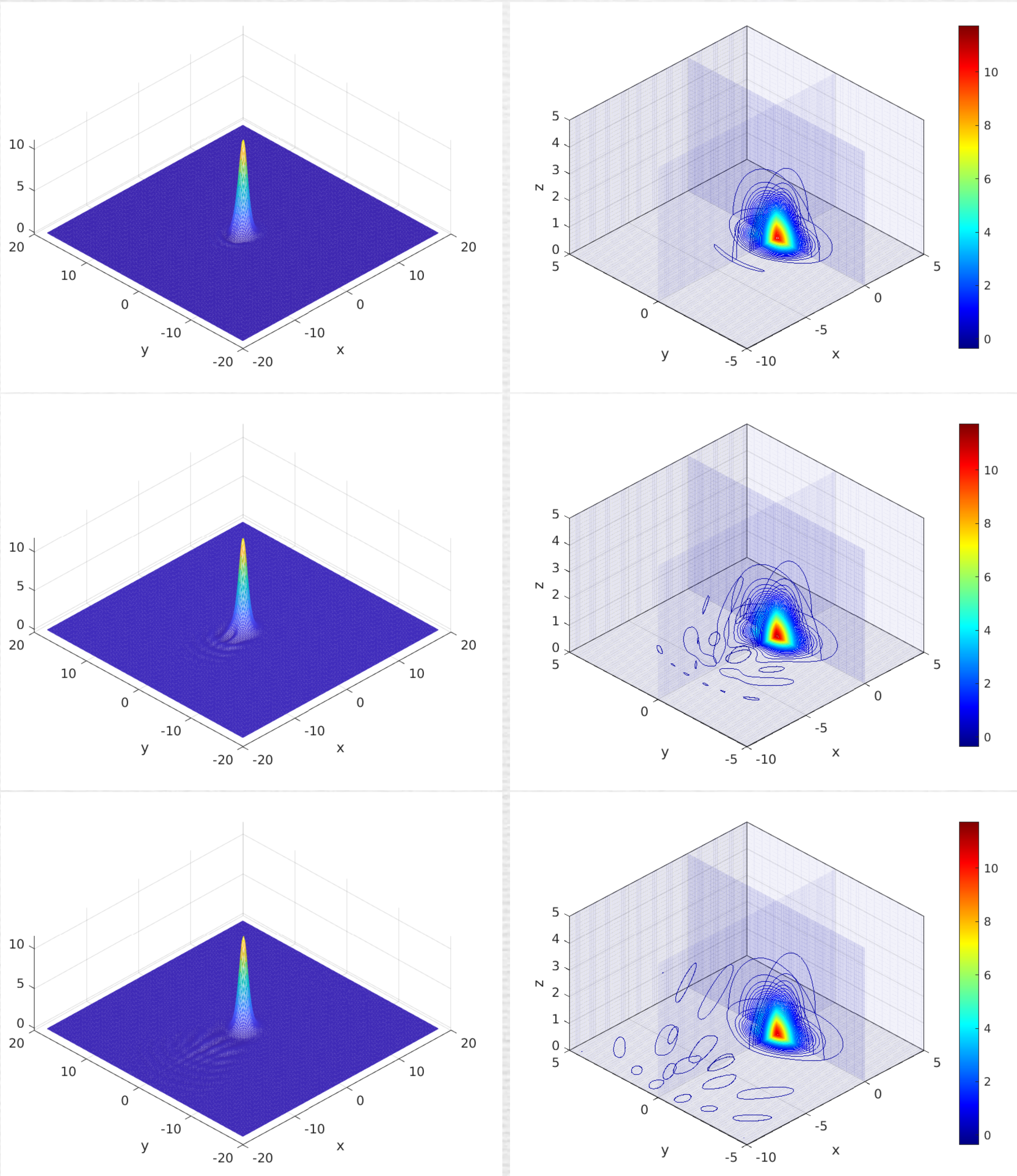




$$u(x, y, z, 0) = 10 \exp(-(x^2 + y^2 + z^2))$$

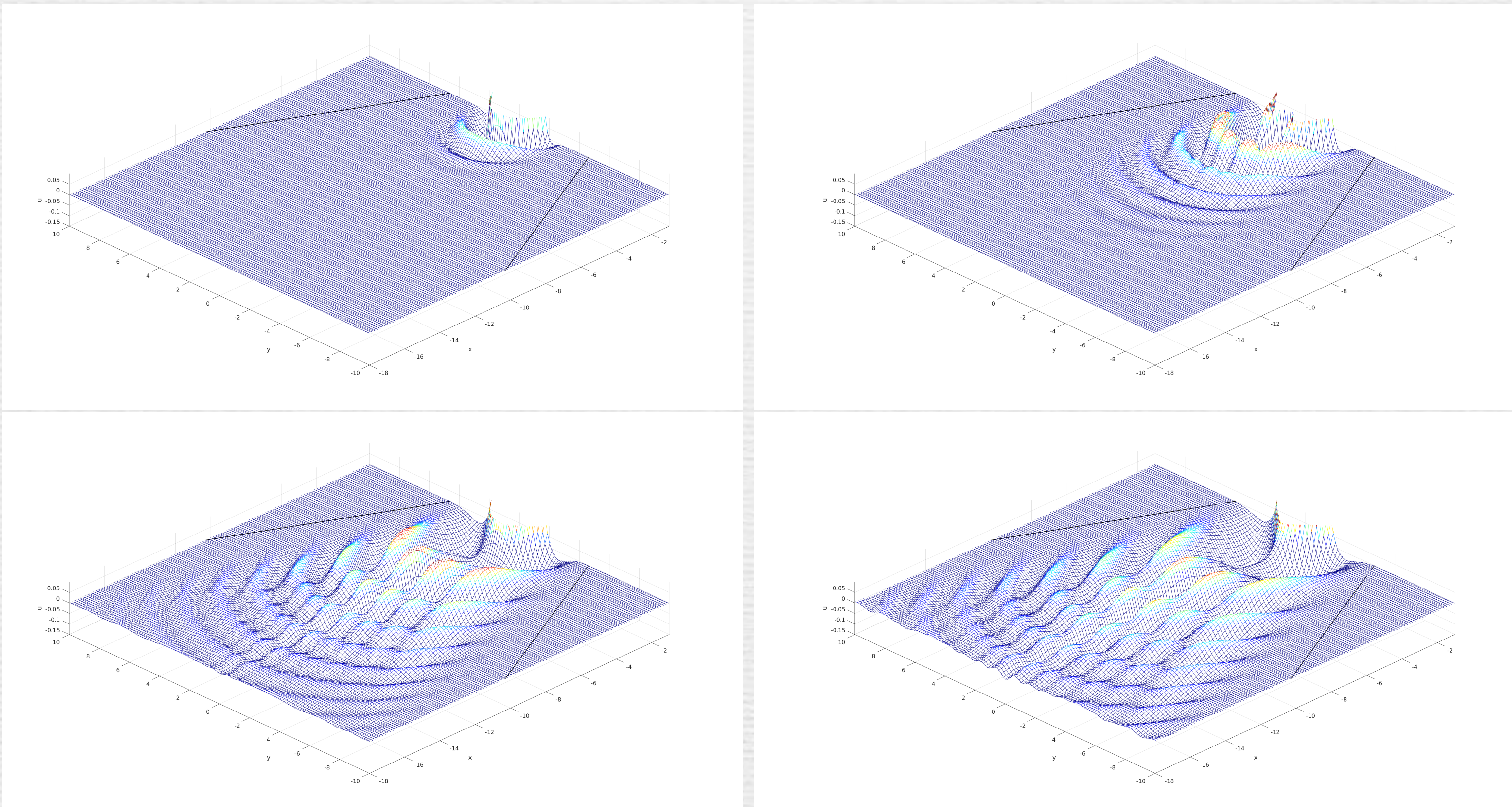
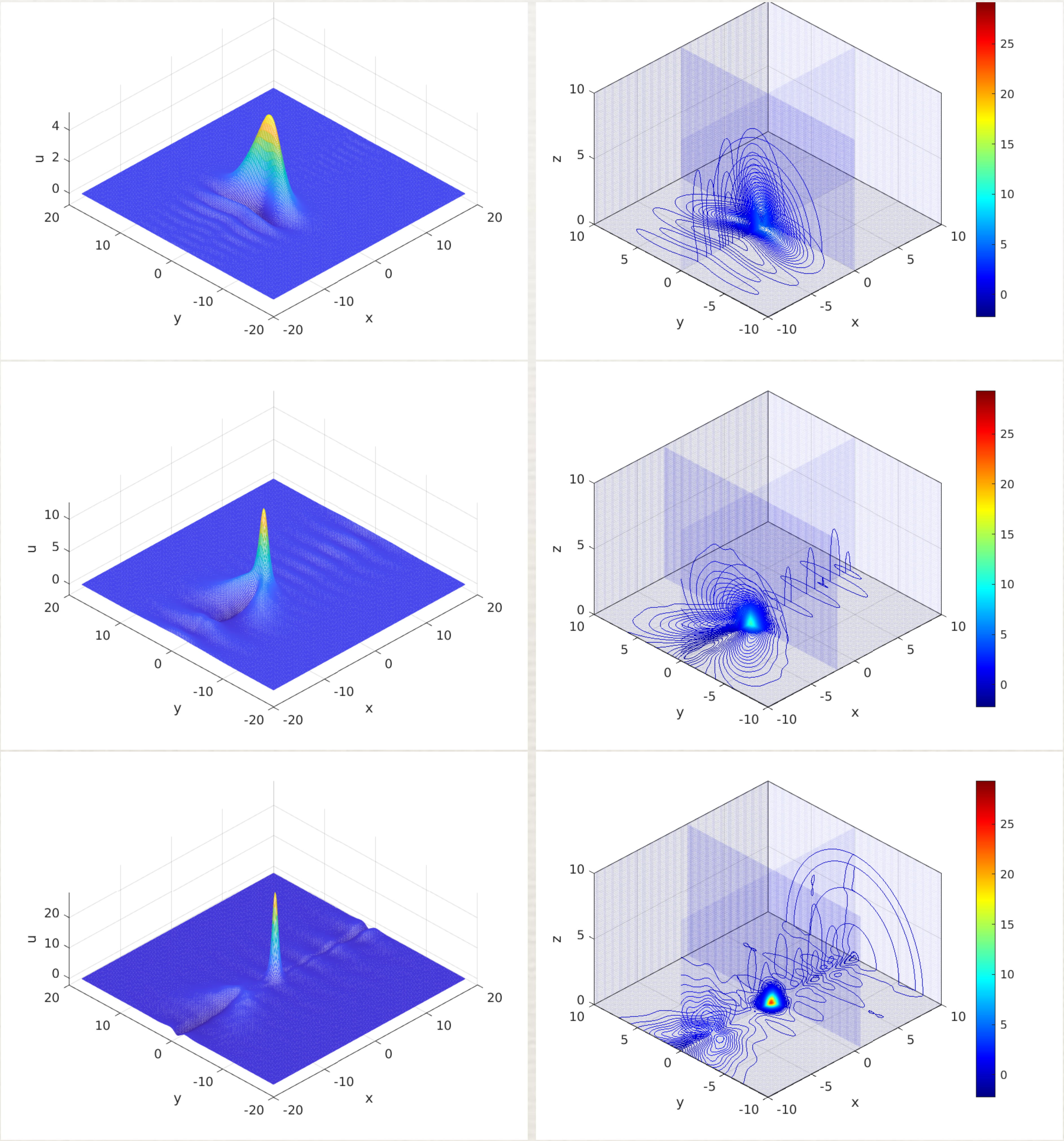
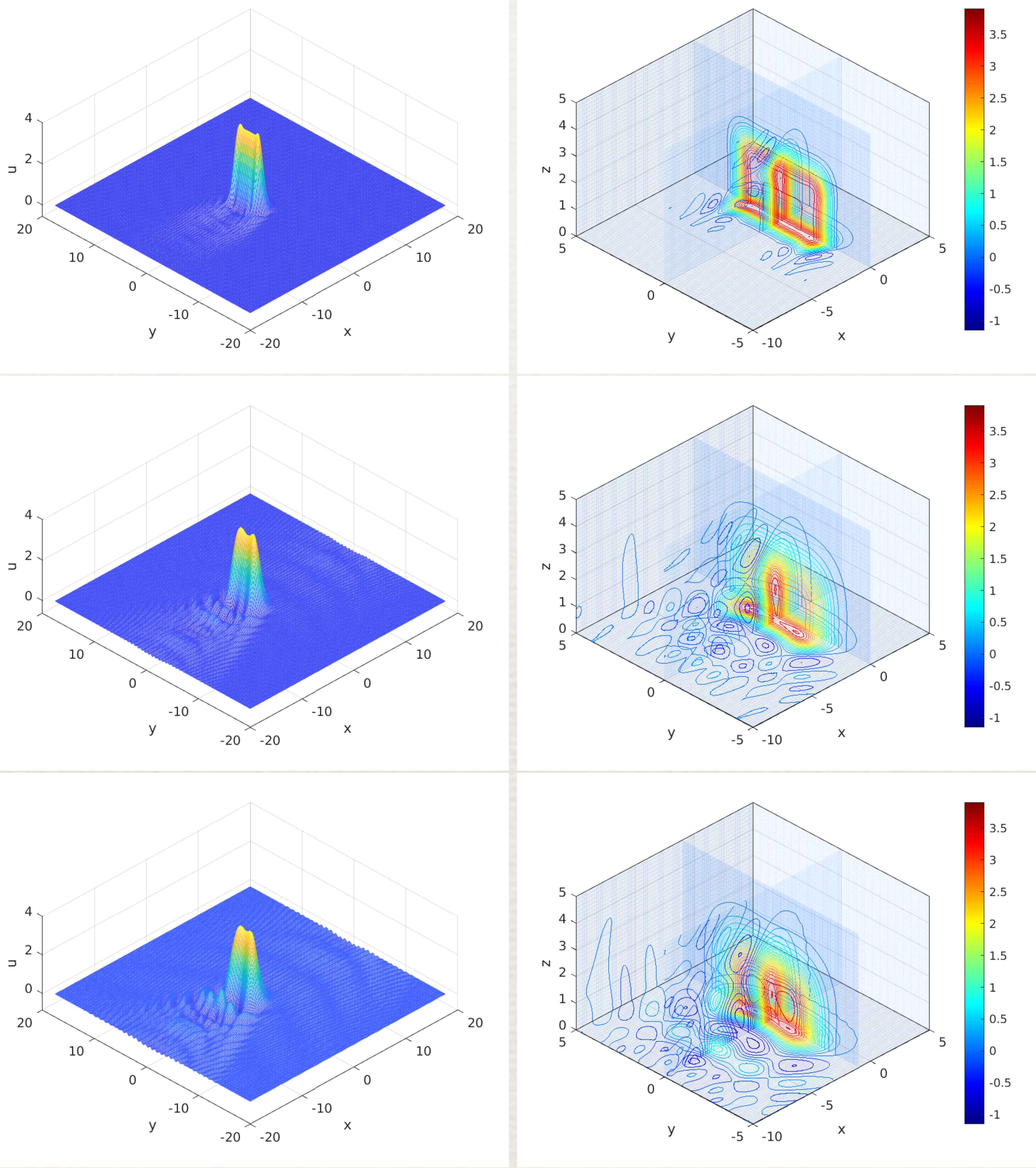


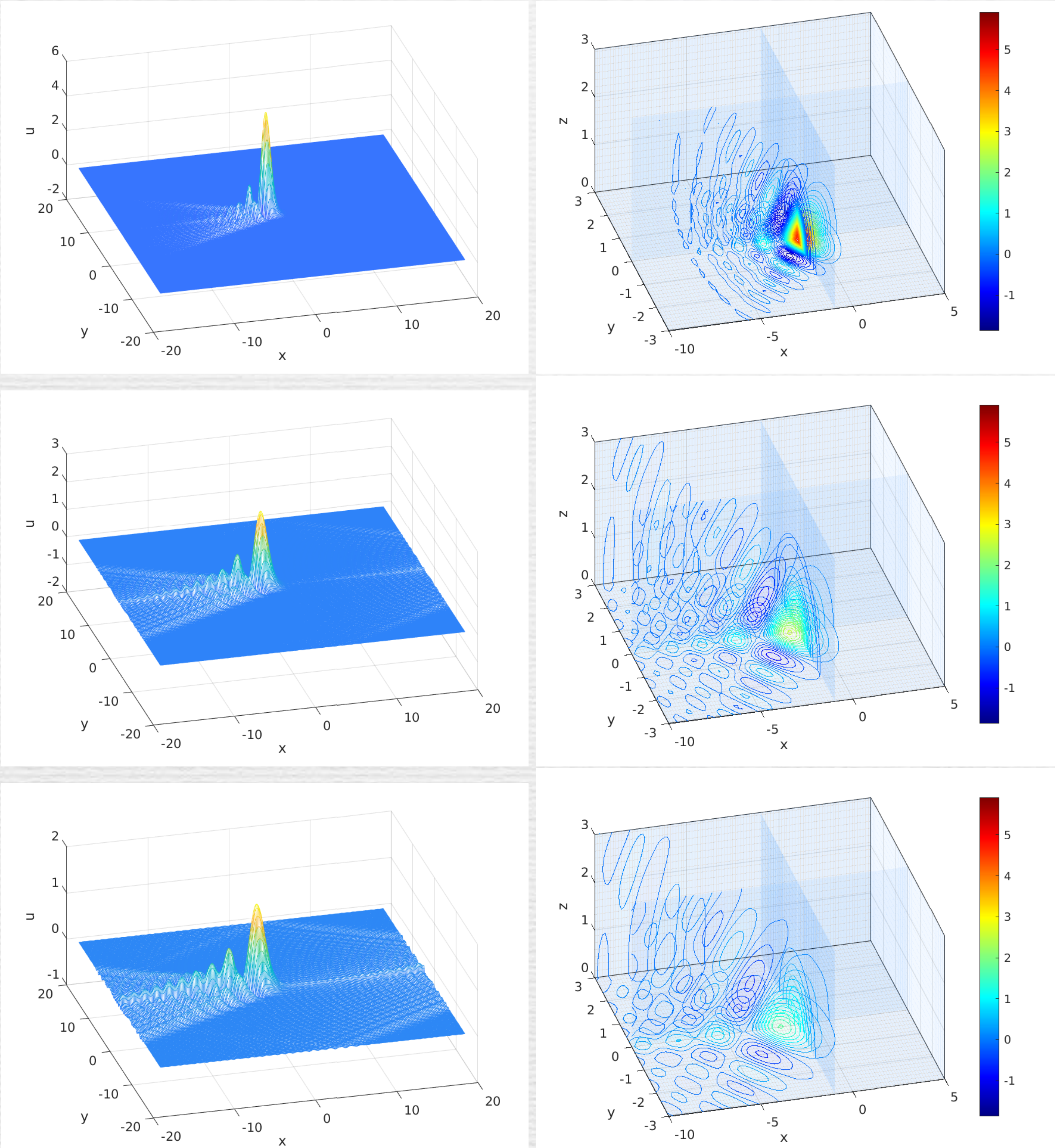
FIGURE 9. Detail view of the radiation developed in the ZK solution with Gaussian initial data $u_0 = 10\, e^{-(x^2+y^2+z^2)}$ at $t = 0.05, 0.15, 0.35, 0.5$. Two dimensional projections onto $z = 0$ with black lines indicating the $\pi/3$ total opening angle of the radiation cone.



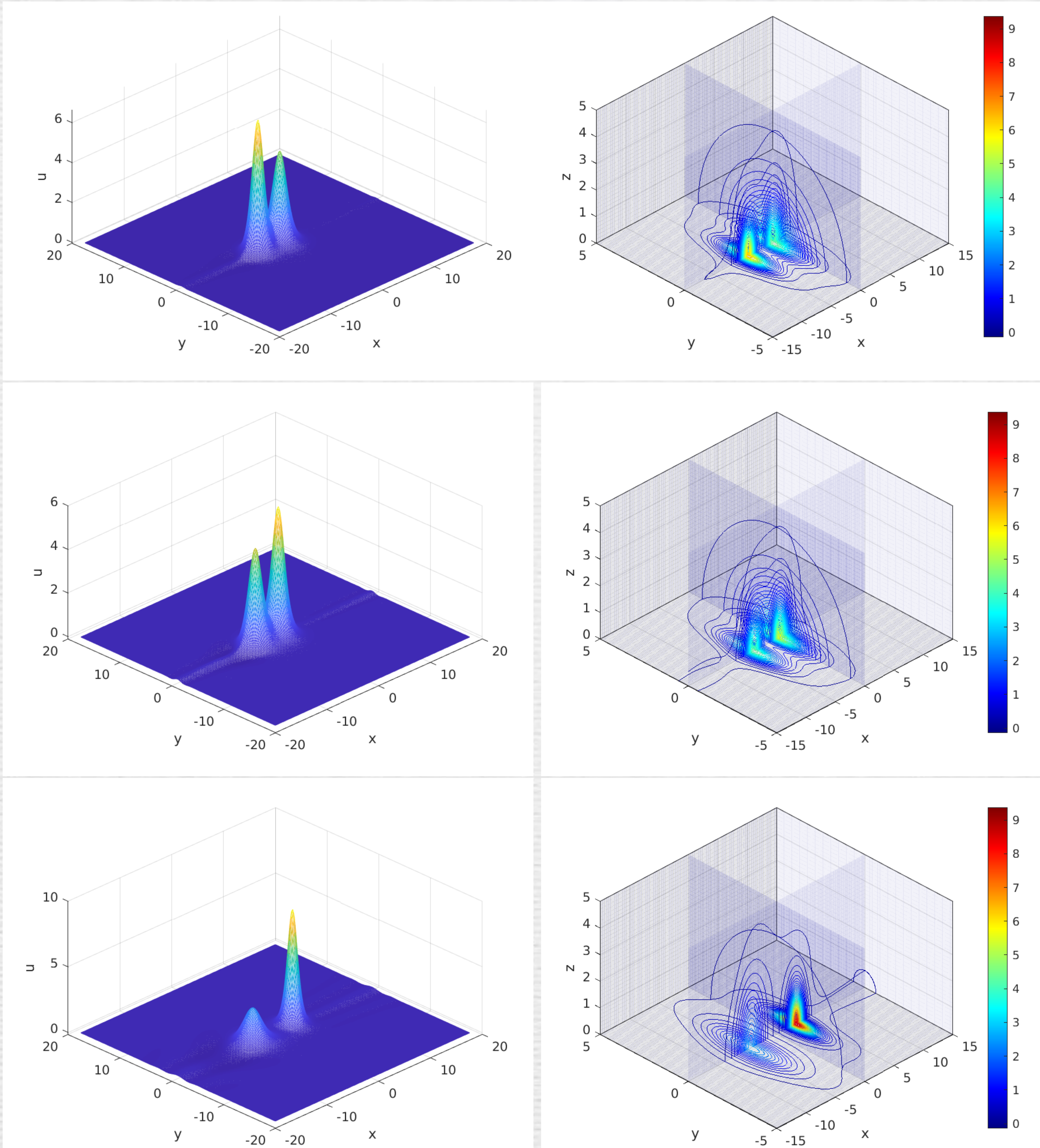
ZK in 3D, p=2 flattened Gaussian, K, Roudenko, Stoilov 2021



ZK in 3D, p=2
wall type,
K, Roudenko,
Stoilov 2021



$$u(x, y, z, 0) = 20/(1 + x^2 + y^2 + z^2)^{10}$$



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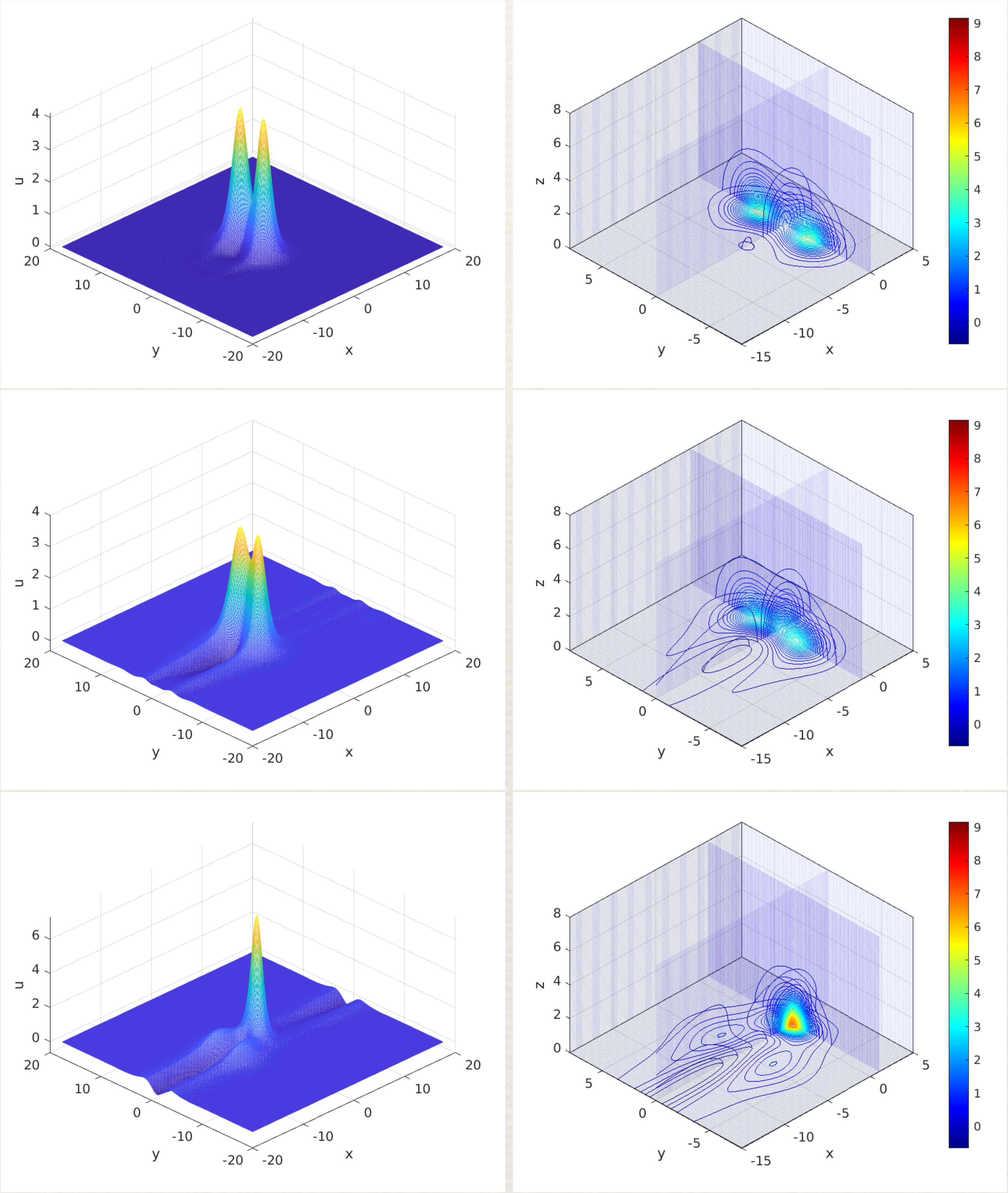
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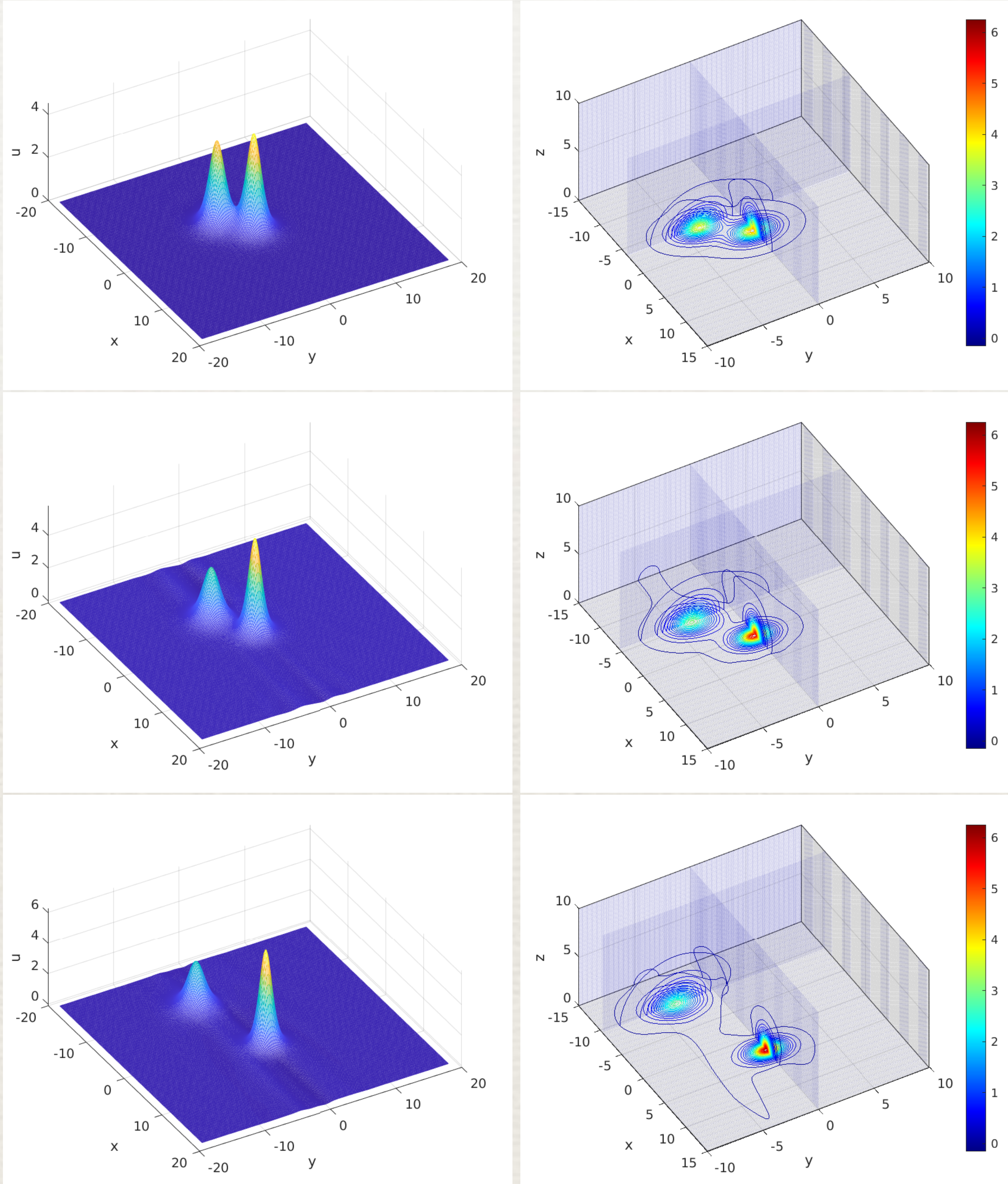

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Outlook

- blow-up computation in 3D
- parallelization on GPUs
- mass critical blow-up as for generalised KdV?
- Supercritical blow-up?

References

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[2] . Klein, J.-C. Saut, N. Stoilov, Numerical study of the transverse stability of line solitons of the Zakharov-Kuznetsov equations, Physica D 448 (2023) 133722

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[4] . Klein, S. Roudenko, N. Stoilov, Numerical study of Zakharov-Kuznetsov equations in two dimensions, J. Nonl. Sci 31(26) (2021) https://doi.org/10.1007/s00332-021-09680-x

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