Rank-1 Lattice Based High-dimensional Approximation and sparse FFT

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joint work with Lutz Kämmerer, Daniel Potts, and Tino Ullrich

- torus $\mathbb{T} \simeq [0,1)$, $\{\mathrm{e}^{2\pi\mathrm{i}kx}\}_{k\in\mathbb{Z}}$ orthonormal basis of $L_2(\mathbb{T})$
- function $f \in L_2(\mathbb{T})$, $f(x) = \sum_{k\in\mathbb{Z}} \hat{f}_k\, \mathrm{e}^{2\pi\mathrm{i}kx}$, $\hat{f}_k = \int_{\mathbb{T}} f(x)\, \mathrm{e}^{-2\pi\mathrm{i}kx}\, \mathrm{d}x \in \mathbb{C}$
- smooth function $f \implies$ fast decay of Fourier coefficients $\hat{f}_k$
- truncated Fourier series $S_I f(x) = \sum_{k\in I} \hat{f}_k\, \mathrm{e}^{2\pi\mathrm{i}kx} \approx f(x)$
- $\hat{f}_k = \int_{\mathbb{T}} f(x)\, \mathrm{e}^{-2\pi\mathrm{i}kx}\, \mathrm{d}x \approx \tilde{\hat{f}}_k := \frac{1}{2N} \sum_{j=0}^{2N-1} f(x_j)\, \mathrm{e}^{-2\pi\mathrm{i}kx_j}$, $x_j := \frac{j}{2N}$

$\Rightarrow$ transfer to multivariate case (tensorization)

$(\tilde{\hat{f}}_k)_{k\in I} \xleftrightarrow[\text{FFT}]{\text{1-dim.}} (f(x_j))_{j=0}^{2N-1}$

$\mathcal{O}(N \log N)$

[Gauß 1866] [Cooley, Tukey 1965]

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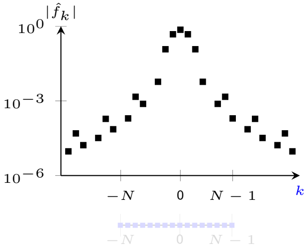

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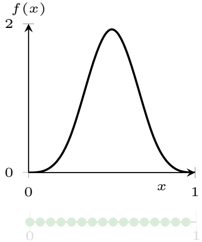

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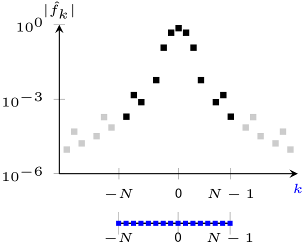

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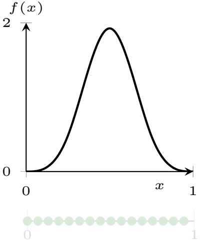

Approximation of periodic functions based on samples

- torus $\mathbb{T} \simeq [0,1)$, $\{\mathrm{e}^{2\pi\mathrm{i}kx}\}_{k\in\mathbb{Z}}$ orthonormal basis of $L_2(\mathbb{T})$
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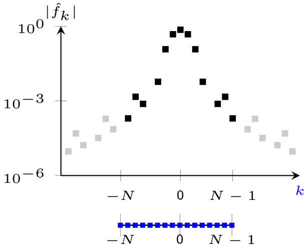
$\Rightarrow$ transfer to multivariate case (tensorization)



$$(\tilde{\hat{f}}_k)_{k\in I} \xleftrightarrow[\text{FFT}]{\text{1-dim.}} (f(x_j))_{j=0}^{2N-1}$$

$$\mathcal{O}(N \log N)$$

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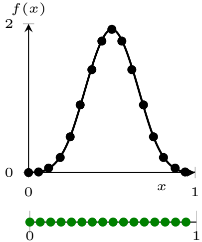

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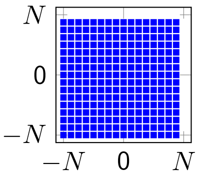
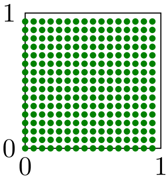
- full grid in frequency domain — equispaced full grid in spatial domain



$$(\tilde{\hat{f}}_{\boldsymbol{k}})_{\boldsymbol{k}\in I} \xleftrightarrow[\text{FFT}]{d\text{-dim.}} (f(\boldsymbol{x}_j))_{j=0}^{|I|-1}$$

$$\mathcal{O}(N^d \log N)$$

curse of dimensionality

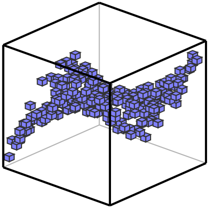


$\Rightarrow$ assumption: sparsity or smoothness first part:

- fast reconstruction of arbitrary high-dimensional trigonometric polynomials $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$ using 1-dimensional FFTs

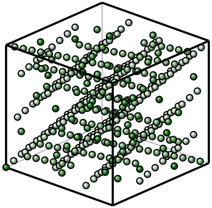
spatial domain:

general known frequency index set $I \subset \mathbb{Z}^d$ multiple rank-1 lattice



$$\left(\hat{p}_{\boldsymbol{k}}\right)_{\boldsymbol{k}\in I} \xleftrightarrow[\text{FFTs}]{\text{1-dim.}} \left(f(\boldsymbol{x}_j)\right)_{j=0}^{M-1}$$

$$\mathcal{O}\left(|I|\,(d+\log|I|)\log^3|I|\right)$$



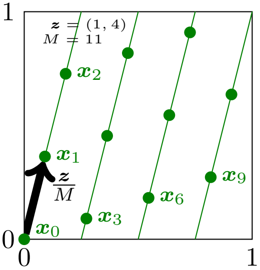
- fast approximation $f(\boldsymbol{x}) \approx \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$ of functions from samples

second part:

- unknown frequency index set I / weights / function space in high dimensions

⇒ dimension-incremental sparse FFT using multiple rank-1 lattices

- $f(\boldsymbol{x}) = p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$, arbitrary freq. index set $I \subset \mathbb{Z}^d$, $|I| < \infty$
- rank-1 lattice $\mathrm{R1L}(\boldsymbol{z}, M) := \{\boldsymbol{x}_j := \frac{j}{M}\boldsymbol{z} \bmod \boldsymbol{1}\}_{j=0}^{M-1}$, $\boldsymbol{z} \in \mathbb{Z}^d$, $M \in \mathbb{N}$, as discretization in spatial domain



Korobov '59
Maisonneuve '72
Sloan & Kachoyan '84,'87,'90
Temlyakov '86
Lyness '89
Sloan & Joe '94
Sloan & Reztsov '01
Li & Hickernell '03
Kämmerer & Kunis & Potts '12

- $f(\boldsymbol{x}) = p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$, arbitrary freq. index set $I \subset \mathbb{Z}^d$, $|I| < \infty$
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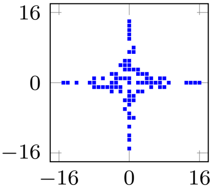
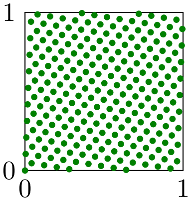
- fast reconstruction of $\hat{p}_{\boldsymbol{k}}$ using 1-dim. FFT? $\hat{p}_{\boldsymbol{k}} \stackrel{?}{=} \frac{1}{M} \sum_{j=0}^{M-1} p_I(\boldsymbol{x}_j)\, \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}_j}$



$$(\hat{p}_{\boldsymbol{k}})_{\boldsymbol{k}\in I} \overset{?}{\longleftarrow} (p_I(\boldsymbol{x}_j))_{j=0}^{M-1}$$

$$\mathcal{O}(M \log M + d\,|I|)$$



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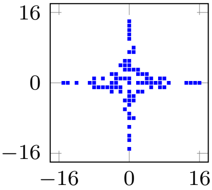
  $\Rightarrow$ reconstruction property: [Kämmerer,Kunis,Potts '12] [Kämmerer '12]
  $\boldsymbol{k}\cdot\boldsymbol{z} \not\equiv \boldsymbol{k}'\cdot\boldsymbol{z} \pmod{M}$ for all $\boldsymbol{k}, \boldsymbol{k}' \in I$, $\boldsymbol{k} \neq \boldsymbol{k}'$



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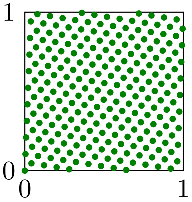

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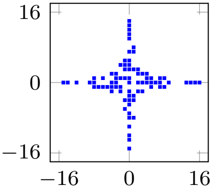
- fast approximation of $f \in L_2(\mathbb{T}^d) \cap C(\mathbb{T}^d)$ using rank-1 lattice sampling error estimates in [Byrenheid,Kämmerer,Ullrich,V. '17] [V. '17]



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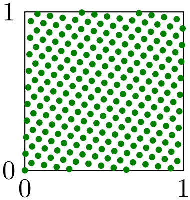
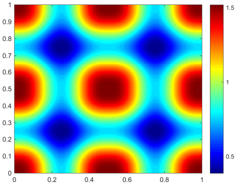
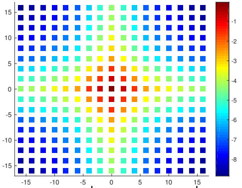


Example – function from Sobolev space of dominating mixed smoothness

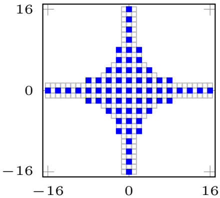
- $f(\boldsymbol{x}) := \prod_{s=1}^{d} \left(2 + \operatorname{sgn}((x_s \bmod 1) - \frac{1}{2})\, \sin(2\pi x_s)^3\right)$,
- hyperbolic cross $I := \{\boldsymbol{k} \in 2\,\mathbb{Z}^d \colon \prod_{s=1}^{d} \max(1, |k_s|) \leq N\}$



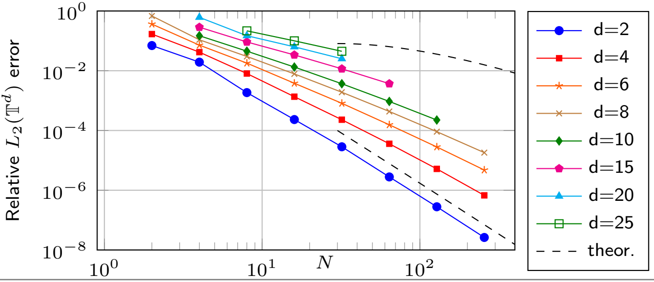
$f\left((x_1, x_2)^\top\right)$



$\log_{10} \left| \hat{f}_{(k_1,k_2)^\top} \right|$



I



- (reconstructing) rank-1 lattice:
  - o number of samples M: $|I| \le M \le |I|^2$, construction: $\mathcal{O}(d\,|I|^3)$, ($|I| \gtrsim N$)
  - + no additional dependence on spatial dimension d in M
  - + very easy and fast computation of Fourier coefficients (single 1-dim. FFT)
  - $\Rightarrow$ two lines of MATLAB code:

```
g_hat = fft(samples)/M;
p_hat = g_hat(mod(I*z',M)+1);
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- improvements? Use more than one rank-1 lattice! (union of several)

$\Rightarrow$ multiple rank-1 lattice sampling [Kämmerer '16] [Kämmerer '17], complexities linear in d, almost linear in sparsity $|I|$ (for $|I| \gtrsim N$):

samples:	$\le C\,|I| \log^2 |I|$ (w.h.p.)
construct lattice:	$\le C\,|I|\,(d + \log|I|) \log^3 |I|$ (w.h.p.)
reconstruction / approximation:	$\le C\,|I|\,(d + \log|I|) \log^3 |I|$

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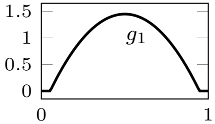
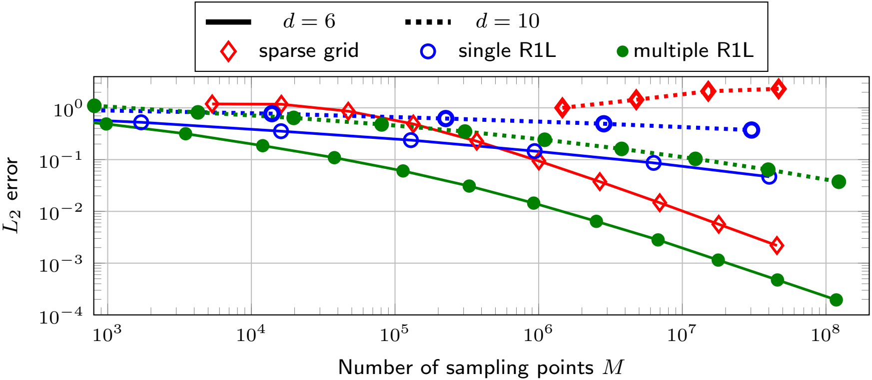
Example — fast approximation

kink function $g_d\colon \mathbb{T}^d \to \mathbb{R}$,

$$g_d(\boldsymbol{x}) = \prod_{s=1}^{d} \left(\frac{5^{3/4} 15}{4\sqrt{3}} \max\left\{ \frac{1}{5} - \left(x_s - \frac{1}{2}\right)^2, 0 \right\} \right)$$



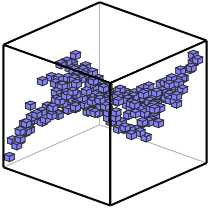


- error estimates for (multiple) rank-1 lattice sampling in [Byrenheid, Kämmerer, Ullrich, V. '17] [V. '17] [Kämmerer, V. '18] first part:

- fast reconstruction of arbitrary high-dimensional trigonometric polynomials $f(\boldsymbol{x}) = p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$ using 1-dimensional FFTs

general known frequency index set $I \subset \mathbb{Z}^d$

spatial domain: multiple rank-1 lattice



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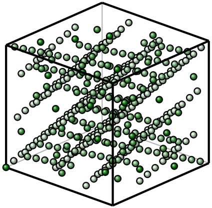

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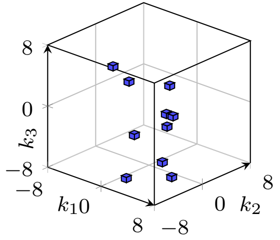


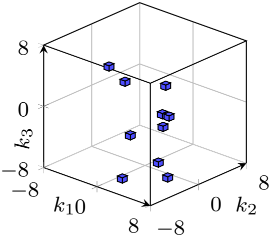
$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$

$$\hat{p}_{k_1} := \frac{1}{17} \sum_{\ell=0}^{16} p\left(\begin{pmatrix} \ell/17 \\ x_2' \\ x_3' \end{pmatrix}\right) \mathrm{e}^{-2\pi\mathrm{i}\frac{\ell k_1}{17}}$$

$k_1 = -8, \ldots, 8$

frequency candidates

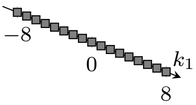
sampling nodes $\{(\frac{\ell}{17}, x_2', x_3')\}_{\ell=0}^{16}$



$I \subset \Gamma = \hat{G}_8^3 := \{-8,-7,\ldots,8\}$

$$\hat{p}_{k_1} := \frac{1}{17}\sum_{\ell=0}^{16} p\left(\begin{pmatrix} \ell/17 \\ x_2' \\ x_3' \end{pmatrix}\right) \mathrm{e}^{-2\pi\mathrm{i}\frac{\ell k_1}{17}}$$

$k_1 = -8, \ldots, 8$

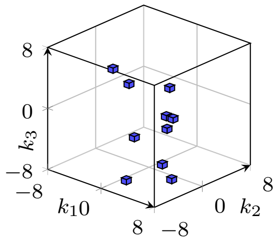


frequency candidates

sampling nodes $\{(\frac{\ell}{17}, x_2', x_3')\}_{\ell=0}^{16}$

High-dimensional dimension-incremental sparse FFT

Method [Potts, V. '15] [V. '17] [Potts, Kämmerer, V. '17], $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$



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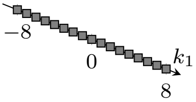
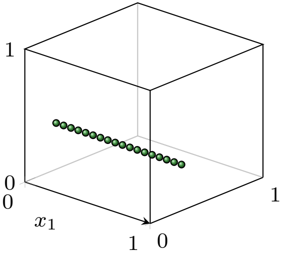
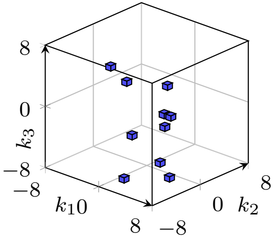


frequency candidates

construct $\longrightarrow$ sampling set



sampling nodes $\{(\frac{\ell}{17}, x_2', x_3')\}_{\ell=0}^{16}$



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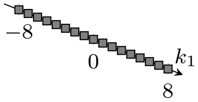
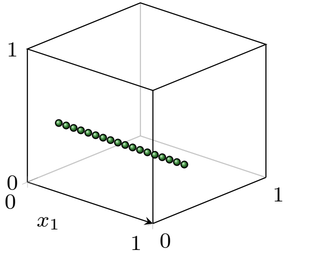
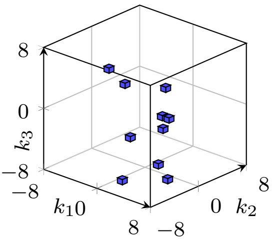


frequency candidates

1-dim.
$\longleftarrow$
FFT



sampling nodes $\{(\frac{\ell}{17}, x_2', x_3')\}_{\ell=0}^{16}$



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$$= \sum_{\substack{(h_2,h_3)\in\{-8,\ldots,8\}^2 \\ (k_1,h_2,h_3)^\top \in \operatorname{supp}\hat{p}}} \hat{p}_{\begin{pmatrix} k_1 \\ h_2 \\ h_3 \end{pmatrix}} \mathrm{e}^{2\pi\mathrm{i}(h_2 x_2' + h_3 x_3')},$$

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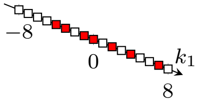
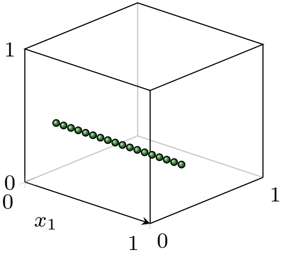
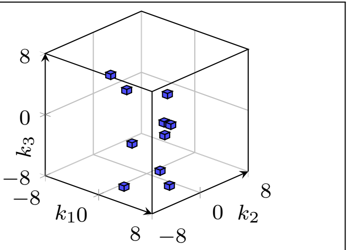


detected frequencies $I^{(1)}$

1-dim. ⟵ FFT



sampling nodes $\{(\frac{\ell}{17}, x_2', x_3')\}_{\ell=0}^{16}$



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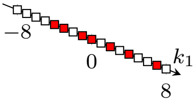
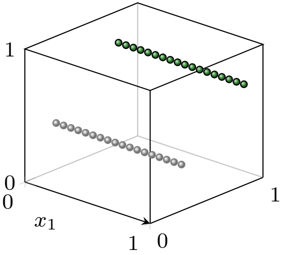


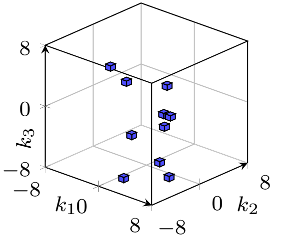
detected frequencies $I^{(1)}$

1-dim. $\longleftarrow$ FFT



sampling nodes $\{(\frac{\ell}{17}, x_2', x_3')\}_{\ell=0}^{16}$

+ repeat (r detection iterations)



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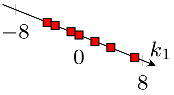
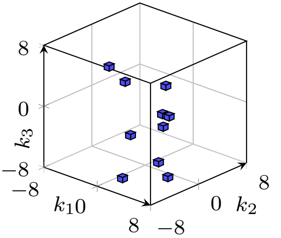


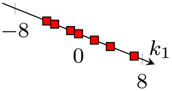
detected frequencies
$I^{(1)}$

frequency candidates

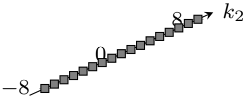
sampling nodes



$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$

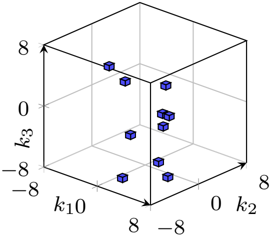


detected frequencies $I^{(1)}$



frequency candidates

sampling nodes



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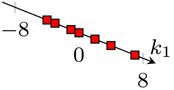
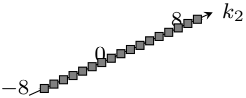
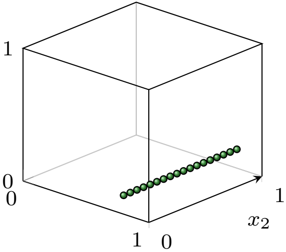


detected frequencies $I^{(1)}$

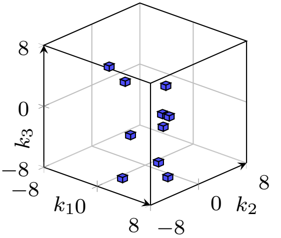


frequency candidates

construct → sampling set



sampling nodes



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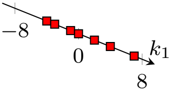
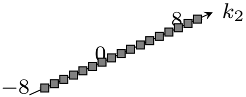
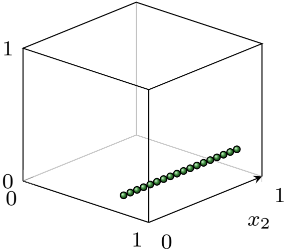


detected frequencies $I^{(1)}$

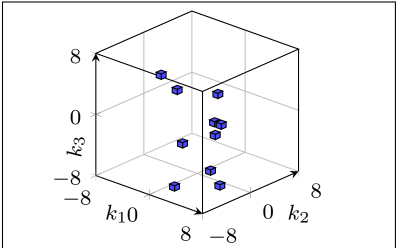


frequency candidates

1-dim. ⟵ FFT



sampling nodes



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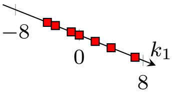
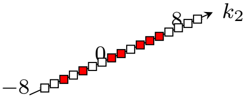
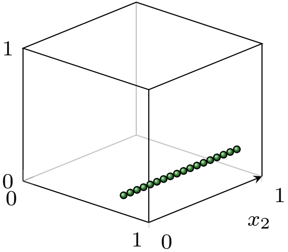


detected frequencies $I^{(1)}$

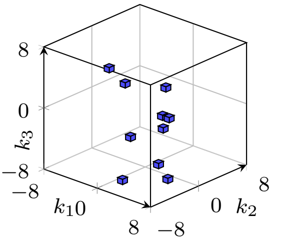


detected frequencies $I^{(2)}$

1-dim. ⟵ FFT



sampling nodes



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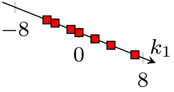
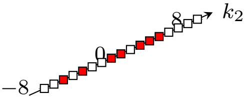
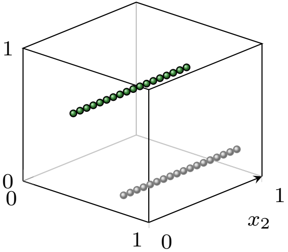


detected frequencies $I^{(1)}$



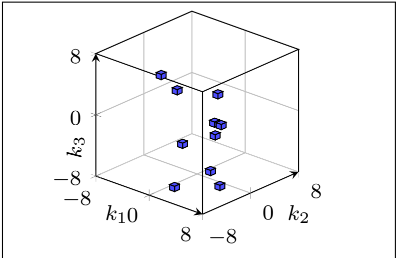
detected frequencies $I^{(2)}$

1-dim. ⟵ FFT



sampling nodes

+ repeat (r detection iterations)



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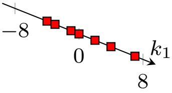
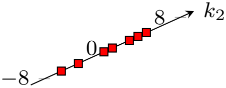
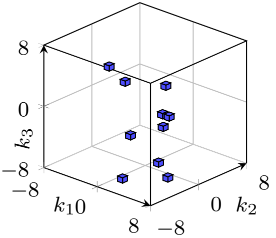


detected frequencies $I^{(1)}$



detected frequencies $I^{(2)}$

frequency candidates $I^{(1)} \times I^{(2)}$



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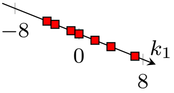
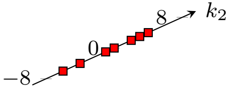
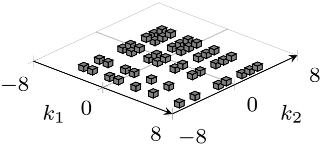


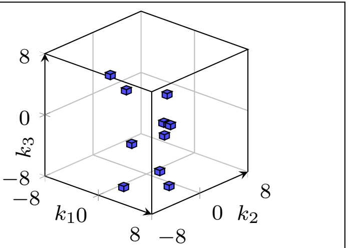
detected frequencies $I^{(1)}$



detected frequencies $I^{(2)}$



frequency candidates $I^{(1)} \times I^{(2)}$



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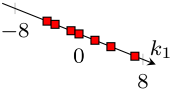
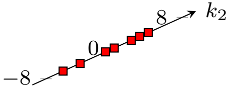
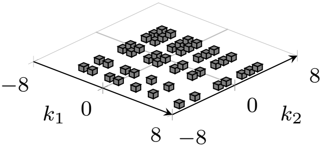


detected frequencies $I^{(1)}$

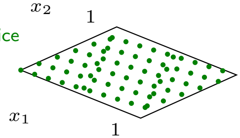


detected frequencies $I^{(2)}$



frequency candidates $I^{(1)} \times I^{(2)}$

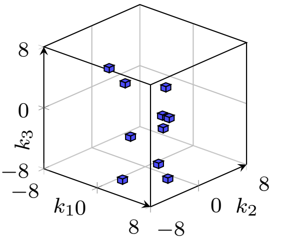
reconstructing
$\longrightarrow$
multiple rank-1 lattice



sampling nodes

High-dimensional dimension-incremental sparse FFT

Method [Potts, V. '15] [V. '17] [Potts, Kämmerer, V. '17], $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$



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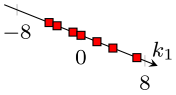
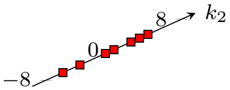
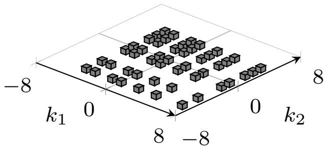


detected frequencies $I^{(1)}$

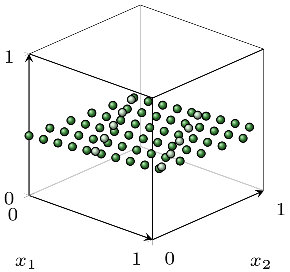


detected frequencies $I^{(2)}$



frequency candidates $I^{(1)} \times I^{(2)}$

1-dim. ⟵ FFTs



sampling nodes

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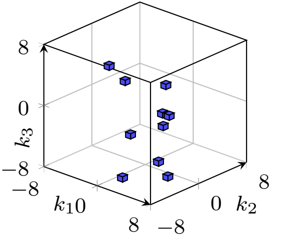

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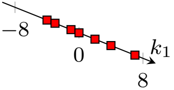
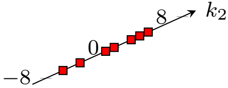
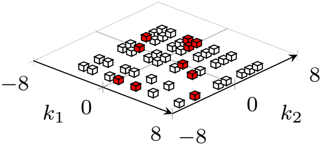


detected frequencies $I^{(1)}$

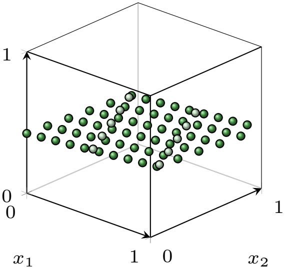


detected frequencies $I^{(2)}$



detected frequencies $I^{(1,2)}$

1-dim. ⟵ FFTs



sampling nodes

High-dimensional dimension-incremental sparse FFT

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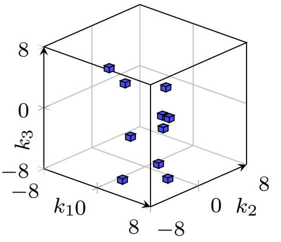

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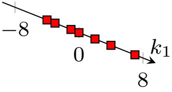
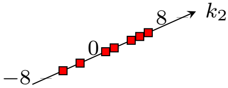
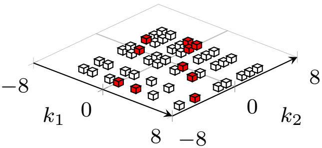


detected frequencies $I^{(1)}$

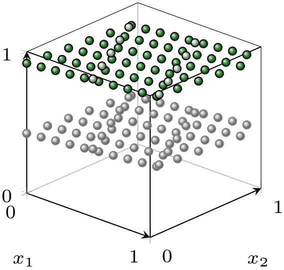


detected frequencies $I^{(2)}$



detected frequencies $I^{(1,2)}$

1-dim. ⟵ FFTs



sampling nodes

+ repeat (r detection iterations)

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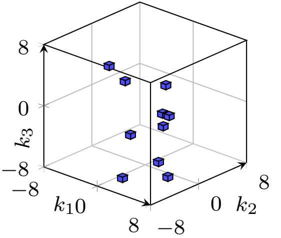

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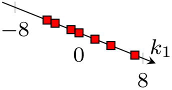
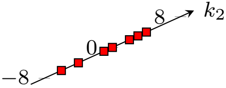
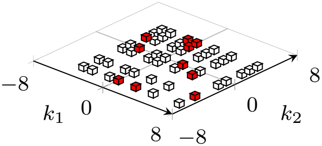


detected frequencies $I^{(1)}$

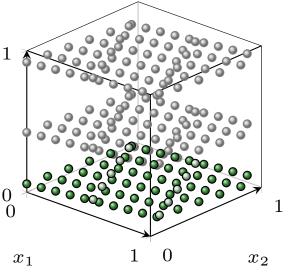


detected frequencies $I^{(2)}$



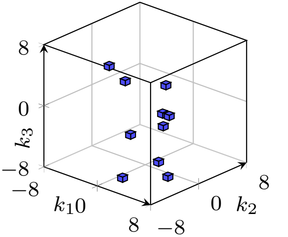
detected frequencies $I^{(1,2)}$

1-dim. ⟵ FFTs



sampling nodes

+ repeat (r detection iterations)



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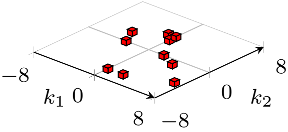
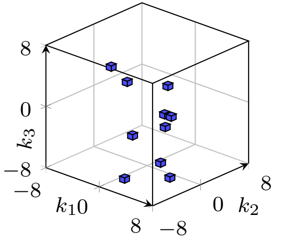


detected frequencies $I^{(1,2)}$

frequency candidates

sampling nodes



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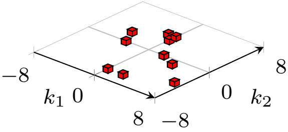

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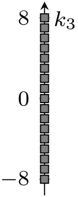
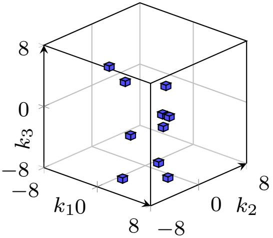


frequency candidates

sampling nodes



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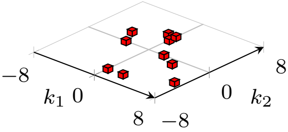

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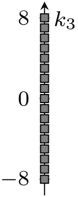
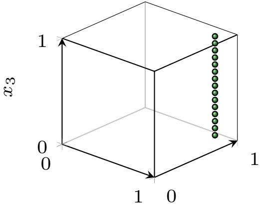
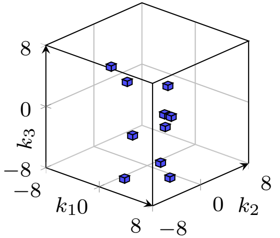


frequency candidates

construct $\longrightarrow$ sampling set



sampling nodes



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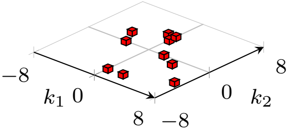

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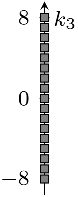
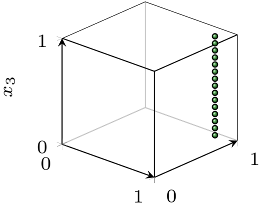
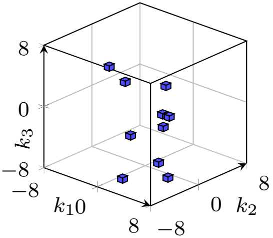


frequency candidates

1-dim. ⟵ FFT



sampling nodes



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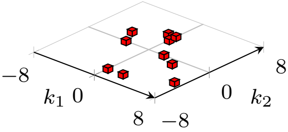
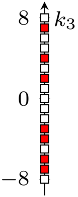
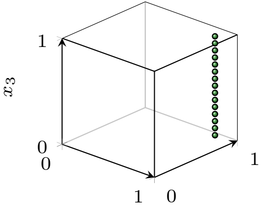


detected frequencies $I^{(1,2)}$



detected frequencies $I^{(3)}$

1-dim. $\longleftarrow$ FFT



sampling nodes

High-dimensional dimension-incremental sparse FFT

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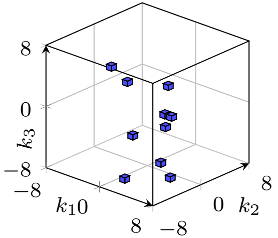

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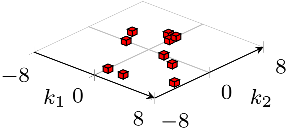
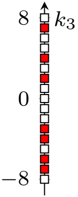
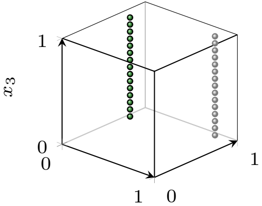


detected frequencies $I^{(1,2)}$



detected frequencies $I^{(3)}$

1-dim.
$\longleftarrow$
FFT



sampling nodes

+ repeat (r detection iterations)

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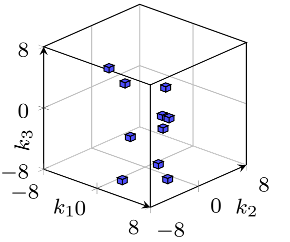

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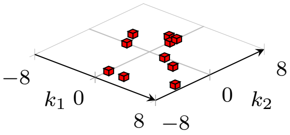
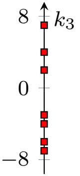


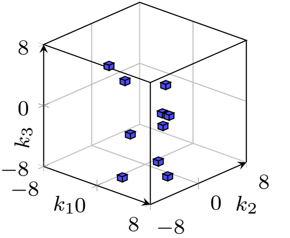
detected frequencies $I^{(1,2)}$



$I^{(3)}$

frequency candidates $I^{(1,2)} \times I^{(3)}$

sampling nodes



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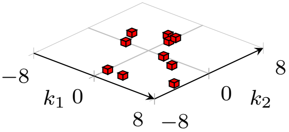
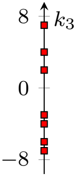
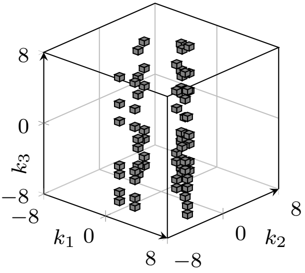


detected frequencies $I^{(1,2)}$



$I^{(3)}$

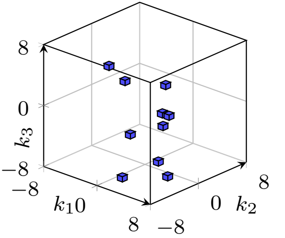


frequency candidates $I^{(1,2)} \times I^{(3)}$

sampling nodes

High-dimensional dimension-incremental sparse FFT

Method [Potts, V. '15] [V. '17] [Potts, Kämmerer, V. '17], $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$



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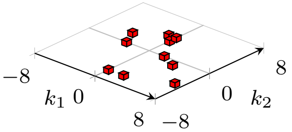

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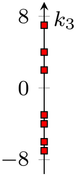
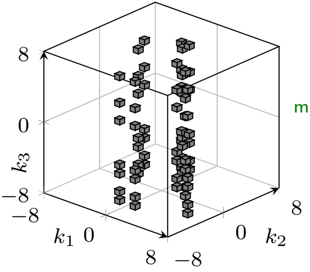
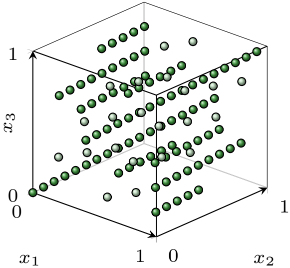


$I^{(3)}$



frequency candidates $I^{(1,2)} \times I^{(3)}$

reconstructing $\longrightarrow$ multiple rank-1 lattice



sampling nodes

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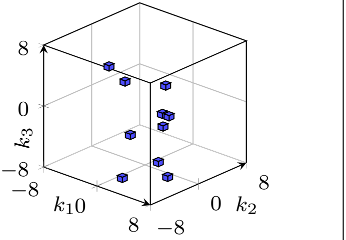

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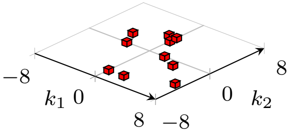
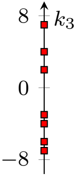
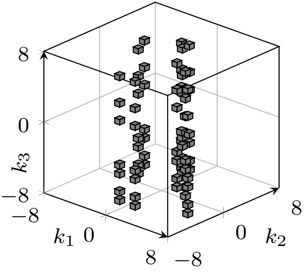


detected frequencies $I^{(1,2)}$

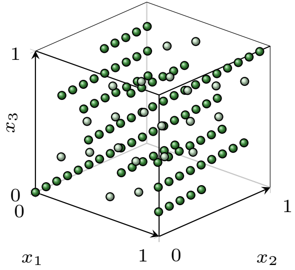


$I^{(3)}$



frequency candidates $I^{(1,2)} \times I^{(3)}$

1-dim. ← FFTs



sampling nodes

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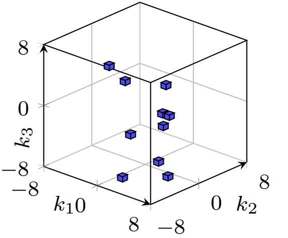

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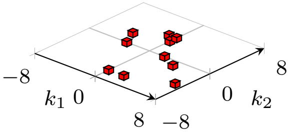
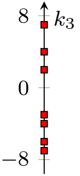
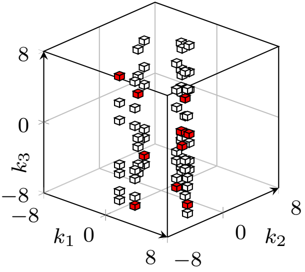


detected frequencies $I^{(1,2)}$

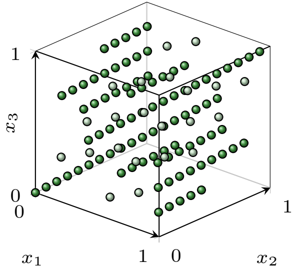


$I^{(3)}$

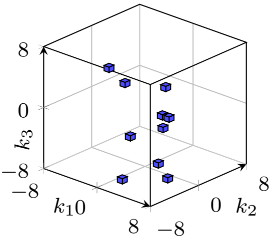


detected frequencies $I^{(1,2,3)} = I$

1-dim. ← FFTs



sampling nodes



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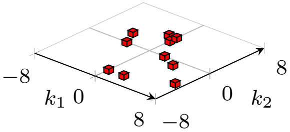

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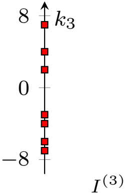

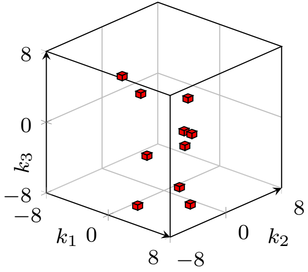

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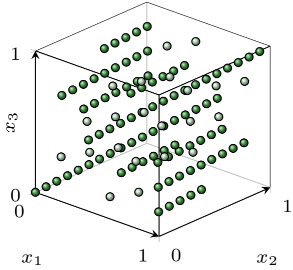
1-dim. ⟵ FFTs



sampling nodes

reconstruction of $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$ with unknown I

using multiple rank-1 lattices:

- sparsity $s = |I|$, search domain $\Gamma = \hat{G}_N^d := \{-N, \ldots, N\}^d \supset I$,

	theory	in practice
samples	$\leq C\, d\, s^2 N \log^3(sN)$ (w.h.p.)	$\leq C\, d\, sN \log^2(sN)$
arithmetic op.	$\leq C\, d^2 s^2 N \log^5(sN)$ (w.h.p.)	$\leq C\, d^2 sN \log^4(sN)$

- MATLAB implementation
- numerically tested for up to 30 spatial dimensions

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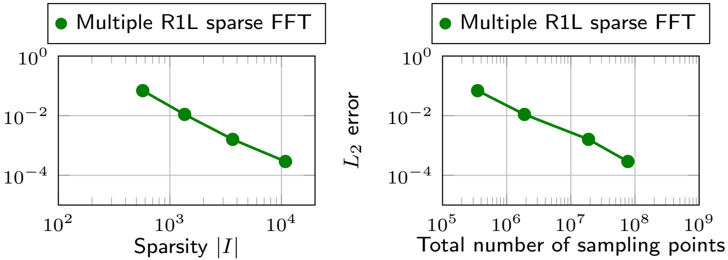
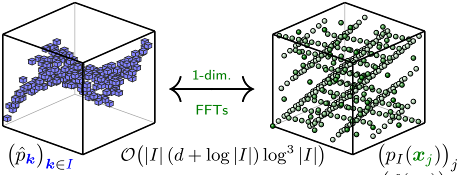
- MATLAB implementation
- numerically tested for up to 30 spatial dimensions

- approximate reconstruction of a function $f \in L_2(\mathbb{T}^d) \cap C(\mathbb{T}^d)$
- $f(\boldsymbol{x}) := \prod_{t\in\{1,3,8\}} B_2(x_t) + \prod_{t\in\{2,5,6,10\}} B_4(x_t) + \prod_{t\in\{4,7,9\}} B_6(x_t)$,
  $B_m(x) = \sum_{k\in\mathbb{Z}} C_m \operatorname{sinc}\left(\frac{\pi}{m}k\right)^m (-1)^k \mathrm{e}^{2\pi\mathrm{i}kx}$
  univariate B-spline of order $m \in \mathbb{N}$
- dimension-incremental sparse FFT for $\Gamma = \hat{G}_{64}^{10}$ ($|\hat{G}_{64}^{10}| \approx 1.28 \cdot 10^{21}$):



Conclusion



- known frequency index set $I \subset \mathbb{Z}^d$, multiple rank-1 lattice
  - fast reconstruction of high-dim. trigonometric polynomials p_I [Kämmerer '16] [Kämmerer '17]
  - fast approximation (error estimates for Sobolev-Hilbert type spaces) [Kämmerer, Potts, V. '15] [Byrenheid, Kämmerer, Ullrich, V. '17] [V. '17] [Kämmerer, V. '18]
- **unknown** $I \subset \mathbb{Z}^d$, sampling along (multiple) rank-1 lattices
  - high-dimensional dimension-incremental sparse FFT [Potts, V. '16] [V. '17] [Kämmerer, V. '17]
  - very good numerical results
    for high-dimensional sparse trigonometric polynomials and
    for high-dimensional functions (non-sparse in frequency domain)
- can be transfered to non-periodic case (tensor product Chebyshev bases)
- see also

L. Kämmerer, D. Potts, T. V. **High-dimensional sparse FFT based on sampling along multiple rank-1 lattices**. *ArXiv e-prints* 1711.05152, Nov. 2017.

L. Kämmerer, T. V. **Approximation of multivariate periodic functions based on sampling along multiple rank-1 lattices**. *ArXiv e-prints* 1802.06639, Feb. 2018.

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- L. Kämmerer, T. V. **Approximation of multivariate periodic functions based on sampling along multiple rank-1 lattices**. *ArXiv e-prints* 1802.06639, Feb. 2018.
- Software: MATLAB toolboxes (for single rank-1 lattices) `https://www.tu-chemnitz.de/~tovo`