Implementation of Sparse FFT with Structured Sparsity

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Motivation

- General m-sparse FFT algorithms do not use additional a priori known information about the signal structure:
  - Iwen (2010, deterministic): $\mathcal{O}\left(m^2 \log^4 N\right)$
  - Iwen (2013, randomized w.h.p.): $\mathcal{O}\left(m \log^4 N\right)$,
  - Plonka, Wannenwetsch, Cuyt, Lee (2018): $\mathcal{O}\left(m^2 \log N\right)$.
- FFT algorithms for signals with short support of length m cannot be generalized to two or more support intervals:
  - Plonka, Wannenwetsch (2016, 2017): $\mathcal{O}\left(m \log N\right)$, $\mathcal{O}\left(m \log m \log \frac{N}{m}\right)$,
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Aim: Find a deterministic FFT algorithm for 2π-periodic frequency sparse functions with more general structures:

- Multiple B-length blocks of frequencies,
- Frequencies generated by evaluating n polynomials of degree d at B points.

Contents

Block Sparse Functions

Consider 2π-periodic f with bandwidth N and energetic frequencies contained in n blocks of length B,

$$\{\omega_j, \omega_j + 1, \ldots, \omega_j + B - 1\} \subset \left\{-\left\lceil \frac{N}{2} \right\rceil + 1, \ldots, \left\lfloor \frac{N}{2} \right\rfloor\right\}.$$

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f is block sparse and of the form

$$f : [0, 2\pi] \to \mathbb{C}, \quad f(x) = \sum_{j=1}^{n} \sum_{k=0}^{B-1} c_{\omega_j + k} e^{i(\omega_j + k)x}$$

with finite Fourier transform $\mathbf{c} = (c_\omega)_{\omega \in \{-\lceil \frac{N}{2} \rceil + 1, \dots, \lfloor \frac{N}{2} \rfloor\}}$.
Energetic Frequency: ω with $c_\omega \neq 0$.

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Energetic Frequency: ω with $c_\omega \neq 0$.

Example ($n = 2,\ B = 3$)

$$\mathbf{c} = (0, \ldots, 0, c_{\omega_1}, c_{\omega_1+1}, c_{\omega_1+2}, 0, \ldots, 0, c_{\omega_2}, c_{\omega_2+1}, c_{\omega_2+2}, 0, \ldots, 0)^T$$

Discrete Fourier Transform (DFT)

Definition (Discrete Fourier Transform)

Let $\mathbf{A} = (A(j))_{j=0}^{M-1} \in \mathbb{C}^M$. Define $\widehat{\mathbf{A}} := \left(\widehat{A}(\omega)\right)_{\omega=-\lceil \frac{M}{2} \rceil+1}^{\lfloor \frac{M}{2} \rfloor} \in \mathbb{C}^M$ by

$$\widehat{A}(\omega) := \frac{1}{M} \cdot \sum_{j=0}^{M-1} \mathrm{e}^{\frac{-2\pi \mathrm{i} j\omega}{M}} \cdot A(j).$$

Runtime of the fast DFT: $\mathcal{O}(M \log M)$.

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Definition (Vector of Equidistant Samples)

For $f : [0, 2\pi] \to \mathbb{C}$ and $M \in \mathbb{N}$ define

$$\mathbf{A}_M = (A_M(j))_{j=0}^{M-1} := \left(f\left(\frac{2\pi j}{M}\right)\right)_{j=0}^{M-1}.$$

Main Idea - Decomposition

- $\mathbf{A}_N = \left(f\left(\frac{2\pi j}{N}\right)\right)_{j=0}^{N-1}$.
- n frequency blocks of length $B \Rightarrow \widehat{\mathbf{A}_N}$ is nB-sparse,

$$\widehat{A_N}(\omega) = \begin{cases} c_\omega & \text{if } \omega \in \bigcup_{j=1}^{n} \{\omega_j, \omega_j + 1, \ldots, \omega_j + B - 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

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- General sparse FFT algorithms only efficient for very sparse functions.
- Approach: Decompose input function into sparser functions and apply sparse FFT algorithm to all of them.

Restriction to the Frequencies Congruent to ν

Definition (Restriction)

Let f be block sparse with n blocks of length B, $u \geq B$, $\nu \in \{0, \ldots, u-1\}$.

$$\widehat{A_N^\nu}(\omega) := \begin{cases} \widehat{A_N}(\omega) & \text{if } \omega \equiv \nu \mod u, \\ 0 & \text{otherwise.} \end{cases}$$

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- $\widehat{\mathbf{A}_N^\nu}$: restriction of $\widehat{\mathbf{A}_N}$ to frequencies $\omega \equiv \nu \mod u$.
- $\widehat{\mathbf{A}_N^\nu}$ is at most n-sparse.
- Applying sparse FFT to $\widehat{\mathbf{A}_N^\nu}$ is fast.
- Restriction to residues agrees well with GFFT.

Block Sparse Case

Let f be 1-block sparse.

- f has frequency support $S := \{\omega_1, \omega_1 + 1, \dots, \omega_1 + B - 1\}$.
- Choose $u \geq B$. Then $|\{\omega \equiv \nu \mod u : \omega \in S\}| \leq 1$ for all $\nu = 0, \dots, u - 1$.
- There is at most one energetic frequency congruent to ν modulo u for each residue ν.

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Let f be n-block sparse.

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- Choose $u \geq B$. Then $|\{\omega \equiv \nu \mod u : \omega \in S\}| \leq n$ for all $\nu = 0, \dots, u - 1$.
- There are at most n energetic frequencies congruent to ν modulo u for each residue ν.

Example: $N = 15$, $n = 2$, $B = u = 3$ ($*$: nonzero entries)

$$\widehat{\mathbf{A}_N} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ * \\ * \\ * \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ * \\ * \\ * \\ 0 \end{pmatrix}$$

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SFFT Algorithm for Block Sparse Functions (FAST) I

- Choose $u \geq B$ as a power of 2.
- Apply sparse FFT algorithm to all u at most n-sparse restrictions $\widehat{\mathbf{A}_N^\nu}$.

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- Required samples using GFFT:

$$\mathbf{A}_{s_k t_l u} = \left(f\left(\frac{2\pi j}{s_k t_l u}\right)\right)_{j=0}^{s_k t_l u - 1} \quad \text{for all } k \text{ and } l$$

- t_l: odd primes s.t. $\frac{N}{nu} \leq \prod_{l=1}^{L} t_l$
- s_k: primes s.t. all $\omega \equiv \nu \mod u$ can be uniquely recovered from mod $s_k, t_1, \ldots, t_L$ for more than $K/2$ s_k.

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- Every energetic frequency found for exactly one residue ν modulo u.
- Accurate coefficient estimates guaranteed.
- Choose the nB most energetic returned frequencies.

SFFT Algorithm for Block Sparse Functions (FAST) II

Input: Sparse function f with n blocks of length B and bandwidth N.

$u = 2^{\lfloor \log_2 B \rfloor + 1}$, $t_1 < \cdots < t_L$ minimal, prime s.t. $\frac{N}{nu} \leq \prod_{l=1}^{L} t_l$, $s_1 > \max(n, t_L)$, $K = 2n\lfloor \log_{s_1} \frac{N}{u} \rfloor + 1$, $s_1 < \cdots < s_K$ minimal, prime.

for $k = 1, \ldots, K$, $l = 0, \ldots, L$ **do**

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for $\nu = 0, \ldots, u-1$ **do**

  Apply n-sparse GFFT to $\widehat{\mathbf{A}_N^{\nu}}$ to obtain $S^{\nu} := \{\omega_1^{\nu}, \ldots, \omega_n^{\nu}\}$ and coefficient estimates $x_{\omega_1^{\nu}}, \ldots, x_{\omega_n^{\nu}}$.

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Output: Choose the nB frequencies from $\bigcup_{\nu=0}^{u-1} S^{\nu}$ with largest magnitude coefficient estimates.

Implementations available in Matlab and C++.

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Runtime and Sampling Complexity

Theorem (B., Iwen, Zhang, 2018)

Let $f \in L^2([0, 2\pi])$ be block sparse with n blocks of length B. The FAST algorithm returns an nB-sparse vector $\mathbf{x} \in \mathbb{C}^N$ of accurate Fourier coefficient estimates with runtime

$$\mathcal{O}\left(\frac{B \cdot n^2 \cdot \log B \log^4 N}{\log^2 n}\right)$$

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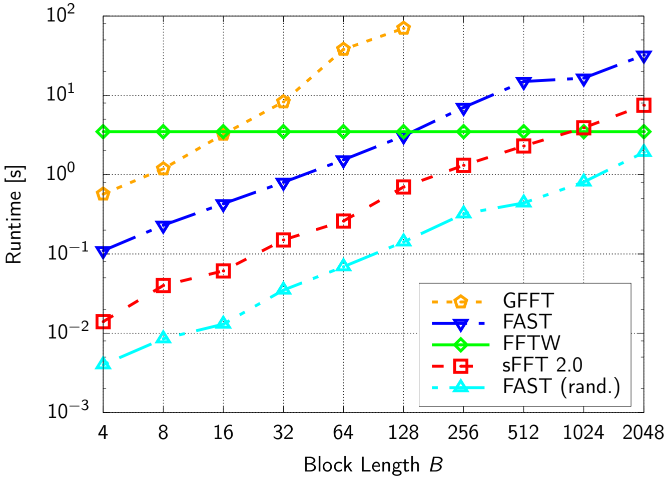
GFFT for nB-sparse functions:

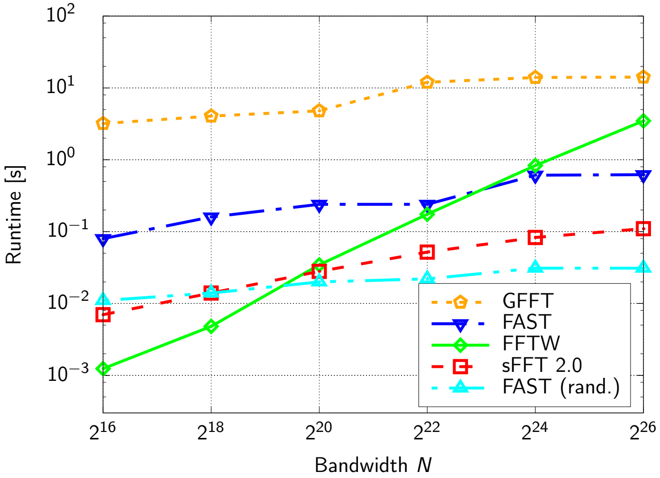
runtime: $\mathcal{O}\left(\frac{(nB)^2 \log^6 N}{\log^2(nB)}\right)$; required samples: $\mathcal{O}\left(\frac{(nB)^2 \log^5 N}{\log^2(nB)}\right)$.

Runtime - Varying the Block Length



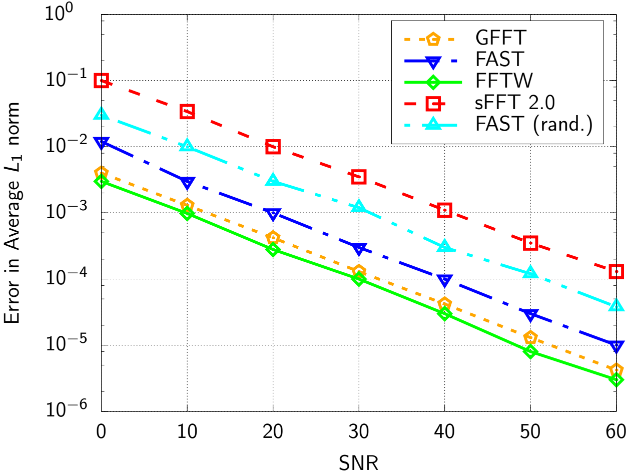
Runtimes of deterministic FFT algorithms for $N = 2^{26}$ and $n = 3$ blocks.

Runtime - Varying the Bandwidth



Runtimes of deterministic FFT algorithms for $n = 2$ blocks of length $B = 64$.

Robustness to Noise



Reconstruction errors of deterministic FFT algorithms for $N = 2^{22}$ and $n = 3$ blocks of length $B = 2^4$.

Generalization of the Technique I

Can more general structures guarantee similar sparsities?

- Block $\{\omega_j, \omega_j + 1, \ldots, \omega_j + B - 1\}$ generated by evaluating $P_j(x) = x + \omega_j$ at $0, 1, \ldots, B - 1$.

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- Are the restrictions $\widehat{\mathbf{A}_N^{\nu}}$ to the frequencies congruent to ν modulo $u > B$ at most nd-sparse?

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Problems:

- $\widehat{\mathbf{A}_N^{\nu}}$ is at most nd-sparse for $\nu \mod u$ if and only if none of the generating polynomials is constant modulo u.
- Knowledge about the polynomial coefficients is hard to obtain.

Generalization of the Technique II

- Choose primes $u_1, \ldots, u_M$ s.t. for more than half of them all restrictions are at most nd-sparse.
- Guaranteed by Chinese Remainder Theorem; related idea used in GFFT.
- Employ median arguments to find correct frequencies and coefficient estimates.

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- Accurate coefficient estimates guaranteed.
- Required samples: $\mathbf{A}_{s_k t_l u_m} = \left(f\left(\frac{2\pi j}{s_k t_l u} \right) \right)_{j=0}^{s_k t_l u_m - 1}$ for all k, l and m.
- Runtime: $\mathcal{O}\left(\frac{Bd^2 n^3 \log^5 N}{\log^2(dn)} \right)$.
- Sampling complexity: $\mathcal{O}\left(\frac{Bd^2 n^3 \log^5 N}{\log B \log^2(dn)} \right)$
- Generalized technique efficient if $B \gg d^2 n \log N$.

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Michigan State University's Sparse FFT Repository - GFFT - Improved sparse Fourier approximation results: faster implementations and stronger guarantees.
https://users.math.msu.edu/users/markiwen/Code.html.

Thank you for your attention.

Main Idea of GFFT

Recovers the most energetic frequencies and accurate estimates for their Fourier coefficients of an m-sparse 2π-periodic function.

- Find smallest primes $t_1, \ldots, t_L$ and $s_1, \ldots, s_K \geq m$ s.t. unique recovery of the frequencies from their residues modulo $s_k, t_1, \ldots, t_L$ is possible by the Chinese Remainder Theorem for all $1 \leq k \leq K$.

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- Required samples: $\mathbf{A}_{s_k t_l} = \left(f\left(\frac{2\pi j}{s_k t_l}\right)\right)_{j=0}^{s_k t_l - 1}$ for all k and l.
- Runtime for m-sparse functions: $\mathcal{O}\left(\frac{m^2 \log^6 N}{\log^2 m}\right)$.
- Sampling complexity for m-sparse functions: $\mathcal{O}\left(\frac{m^2 \log^5 N}{\log^2 m}\right)$.

GFFT Algorithm for Sparse Functions

Input: B-sparse function f with bandwidth N.

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for $k = 1, \ldots, K$, $l = 0, \ldots, L$ **do**

  Compute $\widehat{\mathbf{A}_{s_k t_l}} = \mathbf{DFT}\left(f\left(\frac{2\pi j}{s_k t_l}\right)\right)_{j=0}^{s_k t_l - 1}$.

for $k = 1, \ldots, K$ **do**

  for every residue $h \bmod s_k$ **do**

    Find residues modulo $t_1, \ldots, t_L$ of $\omega \equiv h \bmod s_k$ from $\widehat{\mathbf{A}_{s_k t_l}}$.

  Reconstruct ω from its residues.

for each ω found more than $K/2$ times **do**

  $c_\omega \leftarrow \text{median}\left\{\widehat{A_{s_k t_l}}(\omega \bmod s_k t_l) : k = 1, \ldots, K, l = 1, \ldots, L\right\}$

Output: The B frequencies with largest magnitude coefficient estimates.