Sparse Fourier Transforms, Generalizations, and Extensions

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March 1^{st}, 2019

Work with RuoChuan Zhang & Sami Merhi . . .

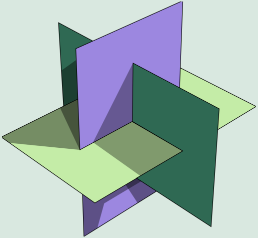
Figure: RuoChuan Zhang (Now @ Research Division of Delphi Automotive), and Sami Merhi (Expected Graduation in Summer 2019)

Compressive Sensing [Candès, Donoho, Tao, ...]

The General Compressive Sensing Framework

Recover $\mathbf{x} \in \mathcal{H}$ from an underdetermined set of linear measurements...

by assuming that it is close to a geometrically simple subset $\mathcal{M} \subset \mathcal{H}$.



Some Fundamental Questions: Which linear measurements (for which $\mathcal{H}$ and $\mathcal{M}$)? What computationally tractable numerical methods exist (for which $\mathcal{H}$ & $\mathcal{M}$)?

- $\mathcal{H} = \mathbb{R}^N$, $\mathcal{M} = \{\mathbf{y} \in \mathbb{R}^N \mid \|\mathbf{y}\|_0 \leq s\}$, $s \ll N$
- $\mathcal{H} = \mathbb{R}^N$, $\mathcal{M} \subset \mathbb{R}^N$ has small Gaussian width, or is a smooth low dimensional submanifold of $\mathbb{R}^N$ with bounded reach, ...
- $\mathcal{H} = \mathbb{R}^{N \times N}$, $\mathcal{M} = \{X \in \mathbb{R}^{N \times N} \mid \mathrm{rank}(X) = s\}$, $s \ll N$
- TODAY: $\mathcal{H} = L^2\left([0, 2\pi]^D, \mathbb{C}\right)$, $\mathcal{M} = \left\{f \in \mathcal{H} \mid \|\hat{f}\|_0 \leq s\right\}$, $s \ll \omega_{\max}$

Where Do Fourier Sparse Signals Appear?

Motivated by

Applications involving wideband signals that are locally frequency sparse in time [see work by Baranuik, Duarte, Hassanie, Tropp, ...].

- Frequency hopping modulation schemes [Lamarr et al., 1941], and wideband spectrum sensing [Hassanie et al., 2014]
- Faster GPS [Hassanieh et. al., 2012]
- Spectral methods for multiscale problems [Daubechies et al., 2007]
- MR Imaging of implicitly sparse specimens [Andronesi et al., 2014]

Notation and Setup

Approximate $f : [0, 2\pi] \mapsto \mathbb{C}$ by a Sparse Trig. Polynomial

$$f(x) \approx \sum_{j=1}^{s} \hat{f}(\omega_j) \cdot \mathbb{e}^{\mathbb{i} x \omega_j} \in \mathcal{M}, \quad \Omega := \{\omega_1, \ldots, \omega_s\} \subset \left(-\frac{N}{2}, \frac{N}{2}\right] \bigcap \mathbb{Z}$$

- In discrete setting we let $f : [0, 2\pi] \mapsto \mathbb{C}$ be the continuous degree $\frac{N}{2}$ trigonometric polynomial interpolant of the given data $\mathbf{f} \in \mathbb{C}^N$.
- We compute point samples, $\mathbf{y} \in \mathbb{C}^m$, with $y_j = f(x_j) + n_j$ for well chosen unequally spaced $x_1, \ldots, x_m \in [0, 2\pi]$.
- The additive evaluation errors, n_j, form the entries of $\mathbf{n} \in \mathbb{C}^m$.
- $\hat{\mathbf{f}} \in \mathbb{C}^N$ contains nonzero entries of $\hat{f}$ for freqs $\in \left(-\frac{N}{2}, \frac{N}{2}\right] \bigcap \mathbb{Z}$.
- $\hat{\mathbf{f}}_s^{\rm opt} \in \mathbb{C}^N$, a best s-term approx. to $\hat{\mathbf{f}} = \mathcal{F}_N \mathbf{f} \in \mathbb{C}^N$ (the DFT of $\mathbf{f}$).

Theorem: A Discrete Result [I., S. Merhi, R. Zhang, 2017]

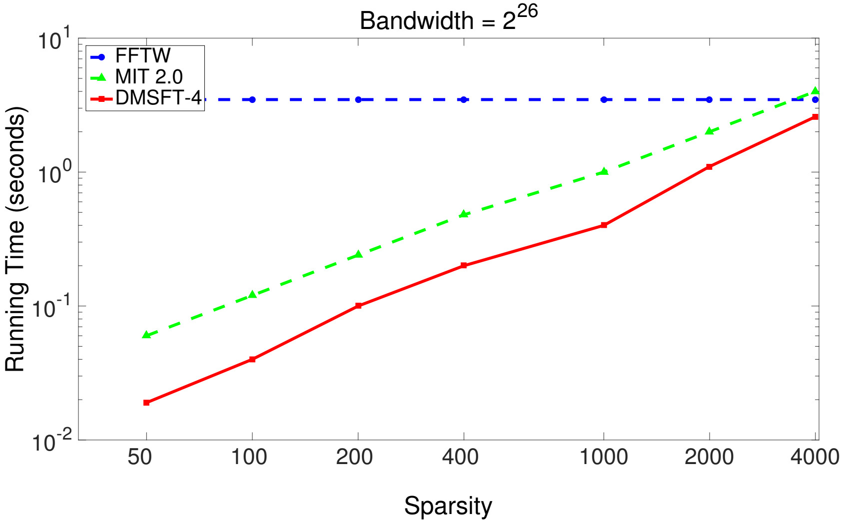
Let $N \in \mathbb{N}$, $s \in [2, N] \cap \mathbb{N}$, $1 \leq r \leq \frac{N}{36}$, and $\mathbf{f} \in \mathbb{C}^N$. There exists an algorithm that will always deterministically return an s-sparse vector $\mathbf{v} \in \mathbb{C}^N$ satisfying

$$\left\| \hat{\mathbf{f}} - \mathbf{v} \right\|_2 \leq \left\| \hat{\mathbf{f}} - \hat{\mathbf{f}}_s^{\rm opt} \right\|_2 + \frac{33}{\sqrt{s}} \cdot \left\| \hat{\mathbf{f}} - \hat{\mathbf{f}}_s^{\rm opt} \right\|_1 + 198\sqrt{s} \, \|\mathbf{f}\|_\infty N^{-r} \qquad (1)$$

in just $\mathcal{O}\left(\frac{s^2 \cdot r^{\frac{3}{2}} \cdot \log^{\frac{11}{2}}(N)}{\log(s)} \right)$-time when given access to $\mathbf{f}$. If returning an s-sparse vector $\mathbf{v} \in \mathbb{C}^N$ that satisfies (1) for each $\mathbf{f}$ with probability at least $(1 - \delta) \in [2/3, 1)$ is sufficient, a Monte Carlo algorithm also exists which will do so in just $\mathcal{O}\left(s \cdot r^{\frac{3}{2}} \cdot \log^{\frac{9}{2}}(N) \cdot \log\left(\frac{N}{\delta}\right) \right)$-time.

- **Proof Idea:** Convolve the trig. polynomial interpolant of **f** with a well chosen periodic Gaussian, and then apply $\mathcal{A}$ from the previous theorems for inf. dim. setting [I., 2013] to the resulting function g.

Publicly Available Codes: Fixed $N = 2^{26}$



- https://sourceforge.net/projects/aafftannarborfa/

Basic Idea of [I., 2013] in the case $\|\mathcal{F}_N \mathbf{f}\|_0 = 1$

- **Example:** $\mathcal{B} \in \{0,1\}^{5\times 6}$, $\mathcal{F}_6 \mathbf{f} \in \mathbb{C}^6$ contains 1 nonzero entry.

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- Reconstruct entry index via Chinese Remainder Theorem
- Two estimates of the entry's value

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- We only utilize 4 entries from $\mathbf{f} \in \mathbb{C}^6$
- Computed Efficiently using 2 FFTs
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Extensions: Compressed Sensing for Parametric PDE

- **Setup:** Given PDE $A(\mathbf{x})u = g$, $\mathbf{x} \in [0, 2\pi]^D$ parameters, approximate Quantity of Interest (QoI) $f(\mathbf{x}) = Gu(\mathbf{x})$ (real valued) as a function of $\mathbf{x}$.
- **Core observation:** QoI $f(\mathbf{x})$ is approximately sparse in appropriate (truncated) product basis T

$$f(\mathbf{x}) \approx \sum_{\mathbf{n} \in \Omega} c_{\mathbf{n}} T_{\mathbf{n}}(\mathbf{x})$$

  that is, each $\mathbf{n} \in I_D := \{0, \ldots, N-1\}^D$, indexes a basis function $T_{\mathbf{n}}$ and for $\mathbf{n} \in \Omega \subset I_D$ with $s = |\Omega|$ small, $c_{\mathbf{n}} \in \mathbb{C}$ is the coefficient.
- More concretely, we consider basis functions, indexed by $\mathbf{n} \in I_D$, of the form

$$T_{\mathbf{n}}(\mathbf{x}) = \prod_{j=1}^{D} T_{j;n_j}(x_j)$$

  where each $T_{j;n_j}$ is a 1-dim basis function (e.g., $T_{j;n_j}(x) := \mathrm{e}^{\mathrm{i} n_j x}$, orthogonal polynomials, ...).

Extensions: Compressed Sensing for Parametric PDE

- **Recall our goal:** Approximate $f : [0, 2\pi]^D \to \mathbb{R}$ sparse in $\{T_{\mathbf{n}}\}$.
- **Samples:** Each PDE solve yields $\approx f(\mathbf{x}_j)$ for some fixed set of parameters $\mathbf{x}_j$ (of our choosing).
- **In matrix form:** Recover s-sparse $\mathbf{c}$ from

$$\mathbf{f} = \begin{pmatrix} f(\mathbf{x}_1) \\ f(\mathbf{x}_2) \\ \vdots \\ f(\mathbf{x}_m) \end{pmatrix} = \begin{pmatrix} T_{\mathbf{n}_1}(\mathbf{x}_1) & T_{\mathbf{n}_2}(\mathbf{x}_1) & \cdots & \cdots & T_{\mathbf{n}_{N^D}}(\mathbf{x}_1) \\ T_{\mathbf{n}_1}(\mathbf{x}_2) & T_{\mathbf{n}_2}(\mathbf{x}_2) & \cdots & \cdots & T_{\mathbf{n}_{N^D}}(\mathbf{x}_2) \\ \vdots & \vdots & & \ddots & \vdots \\ T_{\mathbf{n}_1}(\mathbf{x}_m) & T_{\mathbf{n}_2}(\mathbf{x}_m) & \cdots & \cdots & T_{\mathbf{n}_{N^D}}(\mathbf{x}_m) \end{pmatrix} \mathbf{c}$$

$$=: \Phi \mathbf{c}$$

- **Strategy [Rauhut, Schwab, Adcock, Webster, ...]:** Ensure, e.g., that $\Phi \in \mathbb{R}^{m \times N^D}$ has the Restricted Isometry Property (RIP) s.t.

$$\max_{\mathcal{S} \subset I_D, |\mathcal{S}| \leq s} \|\Phi_{\mathcal{S}}^* \Phi_{\mathcal{S}} - \mathrm{Id}\|_{2 \to 2}$$

is small. Then, appeal to compressive sensing recovery methods.

Motivation: Compressed Sensing for Parametric PDEs

- **Strategy [Rauhut, Schwab, Adcock, Webster, ...]:**
  - Compute $f(\mathbf{x}_j)$ for $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ (random?)

    Computational cost: $m \times$ (cost of PDE solve).
  - Recover the $\mathbf{c} \in \mathbb{C}^{N^D}$ using ℓ_1 minimization, OMP, CoSaMP, ...

    Computational cost: poly(N^D) – or poly($(\log(N))^D$) using, e.g., hyperbolic cross assumptions to constrain the overall basis size.
- *Prototypical desired result [Rauhut, Schwab, Adcock, Webster, ...]:*

  Recovery guarantees if $m \gtrsim s\,\text{polylog}(N^D, s)$.

The Goal: Approximate $f : [0, 2\pi]^D \mapsto \mathbb{C}$ using as few evaluations as possible, as quickly as possible... in $\mathcal{O}(D^c \ldots)$-time.

Challenge: Can we mitigate *curse of dimensionality* in last step?

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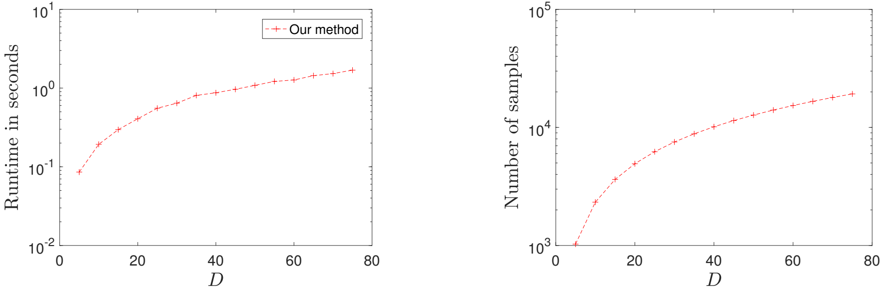
CoSaMP [Needell, Tropp] for General Product Bases

(Recall: $\mathbf{f} = \Phi\mathbf{c}$, $\mathbf{f} \in \mathbb{C}^m$, $\Phi \in \mathbb{C}^{m \times N^D}$, $\mathbf{c} \in \mathbb{C}^{N^D}$ s-sparse)

Algorithm 1 CoSaMP($\Phi, \mathbf{f}, s$) recovery algorithm

$\mathbf{c}^0 = \mathbf{0}$ {Trivial intitial approximation}
$\mathbf{v} \leftarrow \mathbf{f}$ {Current samples=input samples}
$k \leftarrow 0$
repeat
    $k \leftarrow k + 1$
    $\mathbf{w} \leftarrow \Phi^*\mathbf{v}$ {Form signal proxy}
    $\mathcal{S} \leftarrow \mathrm{supp}(\mathbf{w}_{2s})$ {Identify large components}
    $T \leftarrow \mathcal{S} \cup \mathrm{supp}(\mathbf{c}^{k-1})$ {merge supports}
    $\mathbf{a}_T \leftarrow \Phi_T^{\dagger}\mathbf{f}$ {Signal estimation by least-squares}
    $\mathbf{c}^k \leftarrow \mathbf{a}_s^{\mathrm{opt}}$ {Prune to obtain next approximation}
    $\mathbf{v} \leftarrow \mathbf{f} - \Phi\mathbf{c}^k$ {Update current samples}
until halting criterion true

Numerics: Fourier Basis



(a) (b)

Figure: Fourier basis, $N = 20$, $D \in \{5, 10, 15, 20, \cdots, 75\}$, $s = 5$. Reconstruction errors in $\ell^2 \sim 10^{-15}$.

- Standard compressive sensing methods would require more bytes of memory than there are atoms in the universe in order to store their intermediate solutions when $D = 75$...

Thank You! Some other great talks coming up. . .

- **Sina Bittens:** Faster sparse FFTs for functions with structured support. For example, frequencies confined to a few (a priori unknown) bands.
- **Toni Volkmer**, and **Bosu Choi:** More on (Sparse) Fourier transforms in high dimensions!

Post Doc Position Available!

Email if interested (markiwen@math.msu.edu)